

# Deconfined quantum critical points: symmetries and dualities

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The deconfined quantum critical point (QCP), separating the Néel and valence bond solid phases in a 2D antiferromagnet, was proposed as an example of 2+1D criticality fundamentally different from standard Landau-Ginzburg-Wilson-Fisher criticality. In this work we present multiple equivalent descriptions of deconfined QCPs, and use these to address the possibility of enlarged emergent symmetries in the low energy limit. The easy-plane deconfined QCP, besides its previously discussed self-duality, is dual to  $N_f = 2$  fermionic quantum electrodynamics (QED), which has its own self-duality and hence may have an  $O(4) \times Z_2^T$  symmetry. We propose several dualities for the deconfined QCP with  $SU(2)$  spin symmetry which together make natural the emergence of a previously suggested  $SO(5)$  symmetry rotating the Néel and VBS orders. These emergent symmetries are implemented anomalously. The associated infra-red theories can also be viewed as surface descriptions of 3+1D topological paramagnets, giving further insight into the dualities. We describe a number of numerical tests of these dualities. We also discuss the possibility of ‘pseudocritical’ behavior for deconfined critical points, and the meaning of the dualities and emergent symmetries in such a scenario.

## Contents

### I. Introduction

<b>II. Preliminaries and Summary of Results</b>	8
A. Deconfined quantum criticality: $NCCP^1$ and related models	8
B. Fermionic $N_f = 2$ $QED_3$ and related models	12
C. Summary of results	14
<b>III. Easy plane <math>NCCP^1</math> and fermionic <math>N_f = 2</math> <math>QED_3</math></b>	19
A. Justification for the duality	20
B. Symmetries	23
1. Continuous symmetries	24
2. Discrete symmetries	25
C. Allowed symmetry breaking terms	29
<b>IV. <math>SU(2)</math> symmetric <math>NCCP^1</math> and fermionic <math>QED_3</math>-GN</b>	31
A. Duality from the sigma model	32
B. Self-dualities	33
C. Symmetries	34
D. Allowed symmetry breaking terms	35
E. Phase diagram	36
<b>V. Bulk interpretation: <math>U(1) \times U(1)</math> theory</b>	41
<b>VI. Theories with manifest <math>SO(5)</math> symmetry</b>	44
A. Parton construction of $N_f = 2$ $QCD_3$	44
B. Higgs descendent: $N_f = 4$ compact $QED_3$	47
C. Interpretation as surface theory of $3 + 1D$ boson SPT	48
D. Explicit constructions for the $SO(5)$ SPT	51
E. Symmetry-enforced gaplessness	53
<b>VII. <math>QCD_3</math> as the surface of an <math>SO(5)</math> invariant <math>3 + 1D</math> SPT: Formal description</b>	55
A. Gauging $SO(3)_s$	55
B. Gauging $SO(5)$	59
C. Bulk parton construction for boson SPT with $SO(5) \times Z_2^T$ symmetry: Formal derivation	63
D. Chiral spin-liquid	65

<b>VIII. Discussion and Implications of the dualities</b>	66
A. For deconfined criticality	66
B. For QED <sub>3</sub> and QED–GN	68
C. Comparison between the $N = 2$ QED <sub>3</sub> , bilayer honeycomb lattice model, and easy plane spin models	69
<b>IX. Critical and pseudocritical points</b>	70
Acknowledgments	75
A. More precise Lagrangians	76
B. Some other dualities	76
C. A different view of the dualities and emergent symmetries	79
D. Some useful mathematical concepts	83
E. Topological superconductors and the APS theorem	84
References	88

## I. INTRODUCTION

Zeus, the ruler of the Olympian gods, often conceals his identity by changing himself into different forms. Strongly interacting conformal field theories (CFTs), which underlie many different states of matter, can sometimes also be described by Lagrangians with very different forms. In other words, two seemingly different ‘dual’ Lagrangians may correspond to the same CFT. The classic example of such a duality is the equivalence between the 3D  $O(2)$  Wilson-Fisher fixed point and the Higgs transition of bosonic quantum electrodynamics (QED) with one flavor of complex boson [1–3]. Either theory describes interacting lattice bosons at the quantum phase transition between a superfluid phase, in which  $U(1)$  symmetry is spontaneously broken, and a Mott insulating phase, in which it is not.

This paper studies dualities for quantum phase transitions in two spatial dimensions that lie outside the Landau paradigm. Our focus will be on non-Landau transitions between two conventional phases, each of which is well described by a Landau order parameter. The paradigmatic example

of such a phase transition occurs in two dimensional quantum magnets. Square lattice spin-1/2 magnets allow (as the interactions are changed) a conventional Néel antiferromagnetic phase which breaks spin rotation symmetry, and a ‘Valence Bond Solid’ (VBS) phase: a crystal of spin singlets which preserves the spin rotation symmetry while breaking lattice symmetries. A field theory for a putative continuous phase transition between the Néel and VBS phases was described in Refs. [4, 5]. The theory — known as the noncompact  $CP^1$  model ( $NCCP^1$ ) — is formulated in terms of fractionalized ‘spinon’ degrees of freedom (a bosonic field  $z_\alpha$ , with  $\alpha = 1, 2$  an  $SU(2)$  flavor index) coupled to a non-compact  $U(1)$  gauge field  $b$ . Neither the spinon nor the gauge photon however exist as deconfined quasiparticles in either phase. The phase transition has hence been dubbed a ‘deconfined quantum critical point’. Numerical work on specific quantum magnets and related systems [6–22] shows a striking (apparently<sup>1</sup>) continuous phase transition with properties broadly consistent with field theoretic expectations. The deconfined criticality scenario also generalizes to  $SU(N)$  magnets with large  $N$ , where there is a second order phase transition that is under good theoretical control.

Analytic progress on the field theories for  $SU(2)$  deconfined quantum critical points<sup>2</sup> has been challenging. Refs. [26] and [27] showed that a formulation directly in terms of the Landau order parameters for the two phases was possible using a non-linear sigma model, but required the addition of a ‘topological’ term to correctly capture their competition/intertwinement. This term endows the topological defects of each order parameter with nontrivial symmetry properties, enabling the Landau forbidden phase transition. This sigma model formulation gave rise to the possibility that the phase transition may have a large emergent symmetry, which rotates the two Landau order parameters (Néel and VBS) into each other.

Remarkably, recent numerical work finds evidence for the emergence of such a higher symmetry. Specifically a model described by  $NCCP^1$  (a field theory which naively has only  $SO(3) \times U(1)$  symmetry) was seen to have an emergent  $SO(5)$  symmetry at the critical point, at the length scales accessible in the calculations [15]. A good understanding of this emergent extra symmetry is currently not available.

In a different direction, many fascinating new dualities for field theories with  $U(1)$  gauge fields have been found very recently [28–35]. These dualities originated from studies of the surface of

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<sup>1</sup> It is not yet clear whether the transition is truly second order or whether it displays only ‘quasiuniversal’ behavior up to a very large but finite length scale [10–14, 17, 23–25]. We will discuss this in detail in the text.

<sup>2</sup> The continuous global symmetry of the theory is actually  $SO(3) \times U(1)$  and not  $SU(2) \times U(1)$ . By a slight but standard abuse of terminology we will nevertheless refer to it as the  $SU(2)$   $NCCP^1$  model.

three dimensional symmetry protected topological (SPT) phases [28, 29, 31], their relation to three dimensional quantum spin liquids with an emergent  $U(1)$  gauge field [29, 36, 37], and the physics of the half-filled Landau level of two dimensional electrons [38–40]. Other related dualities were discussed in a high energy context (see Ref. [41] and references therein).

From a modern point of view, the classic infra-red duality between the  $O(2)$  Wilson-Fisher theory and bosonic QED is natural because the two field theories have the same global symmetry, namely  $U(1)$ , the same allowed quantum numbers for gauge-invariant local operators (here bosons of integer charge) and the same anomalies (none in this case). The two ultra-violet Lagrangians can therefore be viewed as descriptions of the same physical system with different bare interactions, making it possible that their long distance behaviour is the same. The dualities mentioned above are extensions of this idea to situations in which the operator content and anomalies are nontrivial. In this paper we apply this philosophy to the field theories for deconfined critical points.

We propose and analyze dualities involving these field theories, paying special attention to the realization of symmetries. These dual descriptions give a new way of understanding emergent symmetries relating the Landau order parameters of the Néel and VBS phases.

For the easy-plane version of the  $NCCP^1$  model (describing the Néel-VBS transition in magnets with  $XY$  spin symmetry) several dual descriptions have been discussed in the old and recent literature, as we review below. Some of these dual theories are formulated in terms of bosonic fields while others involve fermionic fields. These boson/fermion fields are coupled to a dynamical  $U(1)$  gauge field. Here we unify these different dual descriptions into a common duality web, and clarify the emergent symmetries of the putative critical fixed point.

For the  $SU(2)$ -symmetric  $NCCP^1$  model, we propose a dual fermionic description as massless  $QED_3$  coupled to a critical real scalar field. We refer to this theory as  $QED_3$ -Gross-Neveu. We show that this duality implies the emergent  $SO(5)$  symmetry at this deconfined critical point (as observed in numerical simulations). We are then led to propose a duality web for the  $SU(2)$  symmetric  $NCCP^1$  theory as well. The existence of this duality web provides an alternate point of view on the emergence of the  $SO(5)$  symmetry at the deconfined critical point.

Remarkably the duality web implies that the  $SU(2)$ -symmetric  $NCCP^1$  model is itself self-dual. Indeed, if we assume this self-duality, the  $SO(5)$  symmetry follows as an inescapable consequence. Conversely the existing evidence for the emergent  $SO(5)$  symmetry strongly supports the conjectured self-duality of the  $SU(2)$ -symmetric  $NCCP^1$  model.

We will show that useful insight into these field theories is obtained by realizing them at the

boundary of 3+1D bosonic Symmetry Protected Topological (SPT) phases. This allows the theories to be regularized in a way which preserves the full internal symmetry of the putative IR fixed point. By contrast, the full internal symmetry of the IR theory cannot be incorporated microscopically in a strictly 2D quantum magnet: it can only be emergent. In field theoretic parlance the symmetry of the IR theory is anomalous, and the anomaly is cancelled when the theory resides at the boundary of a 3+1D SPT phase. Furthermore, in the easy plane case, we show how the ‘bulk’ 3+1D description provides a very simple explanation for the existence of the duality web and the symmetry realizations of the various theories contained therein. For the  $SU(2)$  invariant case, with its putative emergent  $SO(5)$  symmetry, we describe a manifestly  $SO(5)$  invariant formulation in terms of massless fermions coupled to an  $SU(2)$  gauge field, a theory we denote  $N_f = 2$  QCD<sub>3</sub>. This 2+1D theory is shown to have the same anomaly as the proposed  $SO(5)$  invariant fixed point associated with the  $NCCP^1$  theory. This allows us to show that there is a corresponding bulk SPT phase of bosons with global  $SO(5)$  symmetry (i.e. an  $SO(5)$  ‘topological paramagnet’). This boson SPT is characterized in the bulk by its response to an external background  $SO(5)$  gauge field. This response includes a non-trivial discrete theta angle, introduced in Ref. [42], which distinguishes it from a trivial gapped phase of  $SO(5)$ -symmetric bosons. The 2 + 1D theories with anomalous  $SO(5)$  symmetry are alternative descriptions of the surface of this 3 + 1D boson SPT phase.

It is important to distinguish two different versions of statements about duality of quantum field theories that are conflated in the literature. First, there are ‘weak’ duality statements. These assert that the two theories in question have the same local operators, the same symmetries, and the same anomalies (if any). In condensed matter parlance, this means that the two theories ‘live in the same Hilbert space’ and can be viewed as descriptions of the same microscopic system in different limits. These weak dualities are nontrivial statements that can be unambiguously derived. In the context of the present paper their main interest is that they open up the possibility of ‘strong’ dualities. The strong dualities will hold if the putatively dual theories flow, without fine-tuning, to the same nontrivial IR fixed point. For the theories discussed here, it is these strong dualities that would imply the emergence of large, exact symmetries in the infrared. We will not derive the strong dualities in this paper but rather view them as plausible conjectures suggested by the weak dualities.

In fact, the strong dualities can be relevant to the physics up to a very large length scale even in the absence of a true fixed point, if the system shows quasi-universal ‘pseudocritical’ behavior up to a large length scale. We emphasize that it is not yet clear whether the theories we discuss do flow

to nontrivial IR fixed points: this is an ongoing question for numerical work. But for the  $SU(2)$  symmetric  $NCCP^1$  model, simulations show that there is at least apparent critical behavior up to a remarkably large length scale. Numerical evidence for  $SO(5)$  in this regime supports the applicability of the  $SO(5)$  web of dualities. For  $QED_3$ , very recent simulations argued for a flow to a conformal fixed point [43–45], while earlier studies argued for (very weak) chiral symmetry breaking [46]. For the easy-plane  $NCCP^1$  model the current numerical consensus is that the transition is weakly first order. The duality to  $QED_3$  suggests that it may be worth revisiting the Néel-VBS transition in easy plane magnets and related models to look for a second order transition.

We describe consequences of the strong duality conjectures that may be tested in future numerical work. Our proposed duality web for  $SU(2)$ -invariant  $NCCP^1$  and  $QED_3$ –Gross-Neveu involves an emergent  $SO(5)$  symmetry, and leads to clear and testable predictions for the behavior of 2-flavor  $QED_3$  when it is coupled to a critical real scalar field. The web of dualities involving easy-plane  $NCCP^1$  and two-flavour  $QED_3$  are naturally thought of in terms of a ‘mother’ theory with an  $O(4)$  symmetry which rotates the Néel and VBS order parameters. For  $QED_3$  this emergent symmetry should have striking numerically accessible consequences. Our results also show how numerical and analytical studies of  $QED_3$  and  $QED_3$ –GN will provide new information about deconfined criticality.

The duality transformations we employ involve global symmetries with a  $U(1)$  subgroup. For a  $2+1D$  CFT with a global  $U(1)$  symmetry there are two basic formal transformations — denoted  $S$  and  $T$  — which map the theory to other inequivalent theories with a global  $U(1)$  symmetry, assumed also to be CFTs [47, 48]. Our duality transformations can be viewed within this framework. However there are a number of caveats about the standard use of the  $S$  and  $T$  transformations which we discuss in Appendix C. Making standard assumptions about the effect of  $S$  and  $T$  on CFTs allows stronger assertions about deconfined critical points and their symmetries than those discussed above. However it is not clear at this point whether these standard assumptions can be trusted far from the context in which they were originally discussed, i.e. in non-supersymmetric theories that are far from any large  $N$  limit.

## II. PRELIMINARIES AND SUMMARY OF RESULTS

### A. Deconfined quantum criticality: $NCCP^1$ and related models

We first briefly recall the theory of deconfined quantum critical points in quantum magnets. For a spin-1/2 quantum antiferromagnet on a two dimensional square lattice, the transition between the Néel ordered magnet and the Valence Bond Solid (VBS) paramagnet is potentially second order and is described by the  $NCCP^1$  field theory

$$\mathcal{L}_0 = \sum_{\alpha=1,2} |D_b z_\alpha|^2 - (|z_1|^2 + |z_2|^2)^2. \quad (1)$$

Here  $z_\alpha$  ( $\alpha = 1, 2$ ) are bosonic spinons coupled to a dynamical  $U(1)$  gauge field  $b$ , and  $D_{b,\mu} = \partial_\mu - ib_\mu$  is the covariant derivative. (This action and subsequent similar actions are short-hands for the appropriate strongly coupled Wilson-Fisher critical theory where a background gauge field has been promoted to a dynamical field; unless otherwise specified they are written in Minkowski signature.) The model has a global  $SO(3)$  symmetry under which  $z_\alpha$  transforms as a spinor.<sup>3</sup> In the microscopic lattice spin model, this corresponds to the  $SO(3)$  spin rotation. It also has a global  $U(1)$  symmetry associated with the conservation of the flux<sup>4</sup> of  $b$ . In the microscopic lattice spin model, this is not an exact symmetry. Consequently monopole operators (which pick up a phase under a  $U(1)$  rotation) must be added to the Lagrangian. However, it is known that lattice symmetries ensure that the minimal allowed monopole operator (with continuum angular momentum  $\ell = 0$ ) has strength 4. Analytic arguments [4, 5] and numerical calculations [6, 8, 14] strongly support the possibility that these monopoles are irrelevant at the critical fixed point of Eq. (1). The Néel phase is obtained when  $z_\alpha$  is condensed, and the VBS phase when  $z_\alpha$  is gapped. The Néel phase breaks  $SO(3)$  to a  $U(1)$  subgroup while the VBS phase breaks the  $U(1)$  flux conservation symmetry. The Néel order parameter is simply  $\mathbf{N} = z^\dagger \sigma z$  ( $\sigma$  are Pauli matrices), and the VBS order parameter is the strength 1 monopole operator  $\mathcal{M}_b$  which creates  $2\pi$  flux of  $b$ .

If the underlying spin model has only  $O(2)$  ( $XY$ ) spin symmetry — corresponding to conservation of the  $z$ -component of spin, together with a discrete  $\pi$ -rotation of the spins around the  $x$ -axis which

<sup>3</sup> Though  $z_\alpha$  transforms as a spinor under  $SU(2)$ , rotations in the center of  $SU(2)$  can be compensated by a  $U(1)$  gauge transformation so that the spin rotation symmetry of the model is  $\frac{SU(2)}{Z_2} = SO(3)$ .

<sup>4</sup> It is customary in condensed matter literature to call a  $U(1)$  gauge field “non-compact” if its flux is conserved. In high energy literature a  $U(1)$  gauge field is called “compact” if monopole (instanton) operators are local. In this paper we use the condensed matter notation, and call our gauge fields “non-compact”. Of course they are also “compact” in the high energy sense, *i.e* monopole operators can, in principle, be added to the action.



we denote  $\mathcal{S}$  — then the Néel-VBS phase transition is described by the theory

$$\mathcal{L}_{ep-cp1} = \sum_{\alpha=1,2} |D_b z_\alpha|^2 - (|z_1|^4 + |z_2|^4) + \dots \quad (2)$$

This is known as the easy plane  $NCCP^1$  model. In this model the  $XY$  Néel order parameter is  $N_x + iN_y = 2z_1^* z_2$ , while the VBS order parameter is the monopole operator  $\mathcal{M}_b$ . Note that under the  $Z_2$  spin flip symmetry  $\mathcal{S}$  we have

$$\mathcal{S}: \quad z \rightarrow \sigma_x z, \quad b \rightarrow b. \quad (3)$$

Then under  $\mathcal{S}$  the  $XY$  Néel order parameter transforms as  $N_x + iN_y \rightarrow N_x - iN_y$  and the VBS order parameter  $\mathcal{M}_b$  is invariant, as expected microscopically. Later in the paper we will describe the action of time reversal and lattice symmetries for square lattice antiferromagnets (Sec. III B).

The easy-plane theory is known to be self-dual [49], in the sense that it is dual to another easy-plane  $NCCP^1$  theory

$$\mathcal{L}_{ep-cp1-dual} = \sum_{\alpha=1,2} |D_{\tilde{b}} w_\alpha|^2 - (|w_1|^4 + |w_2|^4) + \dots, \quad (4)$$

in which the roles of the two order parameters are switched:  $w_1^* w_2$  is the VBS order parameter, while  $\mathcal{M}_{\tilde{b}}$  is the  $XY$  order parameter. This self-duality is obtained by applying the particle-vortex duality to both spinons:  $z_1 \rightarrow w_2$ ,  $z_2 \rightarrow w_1^*$ . Since the boson mass term is odd under the particle-vortex duality, the self-duality sends  $|z_1|^2 \rightarrow -|w_2|^2$  and  $|z_2|^2 \rightarrow -|w_1|^2$ .

The IR fates of the two  $NCCP^1$  models, and their generalizations with an  $N$ -component spinon field  $z_\alpha$ , have been discussed extensively. They flow to conformal field theories within a  $1/N$  expansion. Directly at  $N = 2$ , numerical calculations see an apparently continuous transition in the  $SU(2)$  invariant  $NCCP^1$  model, but with drifts in some critical properties (which we will discuss in Sec. IX). Further recent studies show the emergence of an  $SO(5)$  symmetry that rotates the Néel and VBS order parameters into one another. For the easy plane case the current wisdom is that the Néel-VBS transition is weakly first order. However, as we will discuss at length, the potential duality with  $QED_3$  may make it interesting to examine this further.

These gauge theories give a natural route to a second order transition between two distinct symmetry broken phases, despite the fact that such a transition is naively forbidden by the Landau theory. In contrast to the standard Landau-Ginzburg-Wilson description, the critical theory is expressed in terms of ‘deconfined’ degrees of freedom (the spinons and the gauge field) which do

not describe sharp quasiparticles in either phase. Physically the breakdown of the Landau paradigm occurs because the topological defects of either order parameter carry non-trivial quantum numbers: the skyrmion defect of the Néel phase carries quantum numbers under lattice symmetries [4, 5, 50, 51], and the vortex defect of the VBS phase carries spin-1/2 [52].

There is an alternative formulation [26, 27] for the competition between the two order parameters directly in terms of a non-linear sigma model. In the  $SU(2)$  invariant case, we define a real 5-component unit vector  $n^a$  ( $a = 1, \dots, 5$ ) such that  $n^{3,4,5}$  correspond to the three components of the Néel vector, and  $n^{1,2}$  to the two real components of the VBS order parameter. The intertwined fluctuations of the two order parameters are then described by an  $SO(5)$  action with a Wess-Zumino-Witten (WZW) term at level 1:

$$S = \frac{1}{2g} \int d^3x (\partial n^a)^2 + 2\pi\Gamma [n^a]. \quad (5)$$

The WZW term  $\Gamma$  is defined in the standard way: the field  $n^a$  defines a map from spacetime  $S^3$  to the target space  $S^4$ , and  $\Gamma$  is the ratio of the volume in  $S^4$  traced out by  $n_a$  to the total volume of  $S^4$ . If  $n^a(x, u)$  is any smooth extension of  $n^a(x)$  such that  $n^a(x, 0) = (0, 0, 0, 0, 1)$  and  $n^a(x, 1) = n^a(x)$ , then

$$\Gamma = \frac{\epsilon_{abcde}}{\text{Area}(S^4)} \int_0^1 du \int d^3x n^a \partial_x n^b \partial_y n^c \partial_t n^d \partial_u n^e. \quad (6)$$

In order to share the symmetry of the  $NCCP^1$  model, the above action must also be supplemented with anisotropy terms that break  $SO(5)$  to  $SO(3) \times U(1)$ . The WZW term correctly captures the non-trivial quantum numbers of the topological defects and is responsible for the non-Landau physics. For example, if the  $U(1)$  symmetry is spontaneously broken, a vortex in the  $U(1)$  order parameter will carry spin-1/2 under the unbroken  $SO(3)$ .

The easy plane case can be obtained from this theory by setting  $n^5 = 0$ . This then leads to an  $O(4)$  non-linear sigma model in  $2 + 1$  spacetime dimensions supplemented with a  $\theta$  term at  $\theta = \pi$ :

$$S = \int d^3x \left[ \frac{1}{2g} (\partial n^a)^2 + \frac{\theta \epsilon_{abcd}}{\text{Area}(S^3)} n^a \partial_t n^b \partial_x n^c \partial_y n^d \right], \quad \theta = \pi. \quad (7)$$

The value  $\theta = \pi$  is robust as a result of the  $\mathbb{Z}_2$  spin-flip symmetry  $\mathcal{S}$  of the easy-plane  $NCCP^1$  model, which changes the sign of  $n_5$  and therefore acts as  $\theta \rightarrow -\theta$ . This topological term is once again responsible for the non-trivial structure of the topological defects.

The sigma model formulation raises the possibility that the phase transition described by Eq. (1) may have an emergent  $SO(5)$  symmetry ( $O(4)$  in the easy plane case). However, we should emphasize that the sigma model is well-defined as a continuum field theory only in the weak coupling

limit, where it is ordered. Here there is a clear semiclassical picture for the effect of the WZW term ( $\theta$  term in the  $O(4)$  case) on the topological defects in the ordered state. For the transition itself — driven by anisotropy for the  $SO(5)$  or  $O(4)$  vector — this ordered state corresponds to a first order phase transition. To study second order Landau-forbidden transitions we need to extend the model to strong coupling, and look for a disordered but power-law correlated  $SO(5)$  invariant fixed point<sup>5</sup>. At strong coupling the sigma model theory is non-renormalizable and requires an alternative formulation as a continuum field theory. Physically, disordered phases of the sigma model (defined with an explicit UV cutoff) correspond to phases where topological defects of the order parameter have proliferated. Thus a modification of the topological defects leads to modifications of the corresponding disordered phases. The sigma model formulation thus exposes the seed, in the ordered phase, of the impending non-Landau physics of the disordered critical regime.

Yet another formulation [26, 27] of the intertwinement of the Néel and VBS orders that maintains manifest  $SO(5)$  symmetry may be obtained by starting with a fermionic spinon description of the square lattice spin-1/2 magnet. This naturally leads to a low energy theory of two flavors of massless Dirac fermions coupled to a dynamical  $SU(2)$  gauge field — a theory we denote  $N_f = 2$   $QCD_3$ . This theory will be useful for some purposes: we discuss it further in Secs. VI, VII.

Finally, deconfined critical field theories also arise in the context of phase transitions between trivial and SPT phases [44, 45, 53]. We will review this connection as needed later in the paper.

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<sup>5</sup> Specifically we need to look for a power-law correlated fixed point which has only one relevant perturbation that breaks  $SO(5)$  to  $SO(3) \times U(1)$  (or rather the symmetry of the lattice model, which is slightly smaller). This perturbation will correspond to the operator whose coefficient is tuned to place the lattice model at its critical point.

### B. Fermionic $N_f = 2$ QED<sub>3</sub> and related models

We now turn our attention to fermionic massless QED<sub>3</sub> models with  $N_f = 2$  flavors of two-component fermions:<sup>6</sup>

$$\mathcal{L}_{qed} = \sum_{j=1}^2 i\bar{\psi}_j \not{D}_a \psi_j + \dots \quad (8)$$

where  $\not{D}_a = \gamma^\mu D_{a,\mu}$  is the gauge covariant Dirac operator that involves a dynamical noncompact  $U(1)$  gauge field  $a_\mu$  (we choose  $\gamma^{0,1,2} = \{\sigma^y, i\sigma_z, i\sigma_x\}$  and  $\bar{\psi} = \psi^\dagger \gamma^0$ ). The flavor symmetry of the model will play an important role in our discussion. We will often, but not always, restrict attention to the case with symmetry under  $SU(2)$  rotations between the two flavors. In addition there is a global  $U(1)$  symmetry associated with the conservation of the flux of the gauge field  $a$ . The theory then has manifest global  $\frac{SU(2) \times U(1)}{Z_2}$  symmetry.<sup>7</sup> It will sometimes however be convenient to consider a more general class of QED<sub>3</sub> theories where the two fermion species are not related by  $SU(2)$  rotations. The theory then only has a global  $U(1) \times U(1)$  symmetry.

By applying the fermion-fermion duality of a single species of Dirac fermion to each of the two fermion species, Refs. [31, 33, 35] demonstrated that, similar to the bosonic easy-plane  $CP^1$  model, this theory is self-dual, *i.e.* it is dual to another  $N_f = 2$  QED

$$\mathcal{L}_{qed-dual} = \sum_{j=1}^2 i\bar{\chi}_j \not{D}_{\bar{a}} \chi_j + \dots \quad (9)$$

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<sup>6</sup> Here we use a more “traditional” procedure of defining a Dirac fermion action. Namely a single Dirac fermion should come together with a Chern-Simons term at level  $k = \pm 1/2$  to avoid a gauge anomaly. A more precise way to define this theory is to use the procedure in Ref. [32, 54] where the partition function of a massless Dirac fermion is written as  $Z_\psi = |Z_\psi| e^{-i\pi\eta[A,g]/2}$ , where  $A$  is the gauge field, either dynamical or background, and  $g$  is the space-time metric.  $\eta$  is defined in terms of eigenvalues of the Dirac operator [54], see Eq. (116). This form corresponds to UV-completing the massless Dirac theory by adding two extra ‘heavy’ Dirac fermions with the same sign of Dirac mass. This enables retaining flavor  $SU(2)$  rotations as an exact symmetry. If we do not care about this  $SU(2)$  symmetry then we can choose the two heavy fermions to have opposite masses in which case the partition function is real. Further strictly speaking  $A$  should be regarded as a  $\text{spin}_c$  connection and not an ordinary  $U(1)$  gauge field, which means that fields with odd charge are fermions [37]. The more precise form of the Lagrangian will be presented in detail in Appendix A.

<sup>7</sup> Note that the fermions themselves transform as spinors under the flavor  $SU(2)$  but rotations by the element of the center  $Z_2$  can be compensated by a  $U(1)$  gauge rotation. This might naively lead to the expectation that the global symmetry is  $SO(3) \times U(1)$ . However the theory has monopole operators which are local and which transform as spinors under the  $SU(2)$ . Rotations by the center of  $SU(2)$  on these can now be compensated by  $\pi$  rotations under the flux conservation *global*  $U(1)$ , and therefore the global symmetry apparent in the Lagrangian is  $\frac{SU(2) \times U(1)}{Z_2}$ .

Given that a particular basis in flavour space had to be selected to perform this duality, we are, strictly speaking, restricting to theories with just a  $U(1) \times U(1)$  symmetry. The dual theory in Eqn. 9 then should also only be taken to have  $U(1) \times U(1)$  symmetry. However, we will later discuss the possibility that with full  $SU(2)$  flavor symmetry this duality survives. As in the easy-plane  $NCCP^1$  model, the roles of the gauge-flux conservation symmetry and the relative phase rotation symmetry between the two Dirac fermions are exchanged in the dual QED theory. The self-duality is obtained by applying the fermionic particle-vortex duality [28, 29, 36, 38] to both flavors of fermions:  $\psi_1 \rightarrow \chi_2$ ,  $\psi_2 \rightarrow \chi_1$ . Since the Dirac mass term is odd under the particle-vortex duality, the self-duality sends  $\bar{\psi}_1\psi_1 \rightarrow -\bar{\chi}_2\chi_2$  and  $\bar{\psi}_2\psi_2 \rightarrow -\bar{\chi}_1\chi_1$ .

The IR fate of  $QED_3$  at  $N_f = 2$  is controversial at this stage. It is not clear whether at low energy the Dirac fermions will spontaneously break the flavor symmetry and gain a mass of the form  $m\bar{\psi}\sigma^z\psi$  — a long-standing issue known as chiral symmetry breaking. Recent numerics [43], however, suggests the possibility that this theory may be stable in the IR (although an earlier study suggests spontaneous chiral symmetry breaking [46]).

We will also be interested in the phases and phase transition of this model when a coupling to an extra scalar  $\phi$  is allowed. The resulting model has the Lagrangian

$$\mathcal{L}_{qed-gn} = \sum_{j=1}^2 i\bar{\psi}_j \not{D}_a \psi_j + \phi\bar{\psi}\psi + V(\phi) \quad (10)$$

Here we have included a potential  $V(\phi) = V(-\phi)$  for the scalar field  $\phi$  (we have suppressed its kinetic term for notational simplicity). The theory is time-reversal symmetric if under time reversal  $\phi \rightarrow -\phi$ . As the potential  $V(\phi)$  is tuned, we expect a phase transition between a time-reversal symmetric phase where  $\langle\phi\rangle = 0$  and a time reversal broken one where  $\langle\phi\rangle \neq 0$ . We will usually refer to Eq. (10) when tuned to this transition as the  $QED_3$ -Gross-Neveu model ( $QED_3$ -GN for short).

Interestingly, with some assumptions, Ref. [27] showed that the low energy behavior of  $N_f = 2$   $QED_3$  was described by the  $O(4)$  sigma model at  $\theta = \pi$  discussed in the previous subsection, again with the proviso that the sigma model needs to be extended to strong coupling. This suggests a connection between the bosonic  $NCCP^1$  theories and the fermionic  $QED_3$  theories. Below we will sharpen this connection through precise duality statements. This will also enable us to understand the emergent IR symmetries of these theories at their critical point.

### C. Summary of results

We now summarize the key results in this paper. We also point out the sections that discuss these statements (and their subtleties) in detail. This section can be viewed as a map of the paper.

1. Both the easy plane and the  $SU(2)$  symmetric  $NCCP^1$  models are part of a web of dualities.

We begin with the easy plane model. It turns out that the easy-plane  $NCCP^1$  model is dual to fermionic QED<sub>3</sub> with  $N_f = 2$  Dirac fermions. This duality was first mentioned in Ref. [33], and will be discussed in detail in Sec. III. As discussed in the literature and reviewed earlier in this section, the two theories also possess their own self-dualities. This leads to a web of four theories, mutually dual to each other, as summarized below:

$$\begin{aligned}
& |D_{b+B}z_1|^2 + |D_{b+B'}z_2|^2 - |z_1|^4 - |z_2|^4 - \frac{1}{2\pi}bd(B+B') - \frac{1}{2\pi}BdB' - \frac{1}{2\pi}B'dB' \\
\iff & |D_{\tilde{b}-B}w_1|^2 + |D_{\tilde{b}+B'}w_2|^2 - |w_1|^4 - |w_2|^4 - \frac{1}{2\pi}\tilde{b}d(B'-B) + \frac{1}{2\pi}BdB' - \frac{1}{2\pi}B'dB' \\
\iff & \bar{\psi}_1 i \not{D}_{a-B} \psi_1 + \bar{\psi}_2 i \not{D}_{a+B} \psi_2 + \frac{1}{2\pi}adB' + \frac{1}{4\pi}(BdB - B'dB') \\
\iff & \bar{\chi}_1 i \not{D}_{\tilde{a}-B'} \chi_1 + \bar{\chi}_2 i \not{D}_{\tilde{a}+B} \chi_2 + \frac{1}{2\pi}\tilde{a}dB + \frac{1}{4\pi}(BdB - B'dB'). \tag{11}
\end{aligned}$$

Here  $b$  and  $\tilde{b}$  are ordinary dynamical  $U(1)$  gauge fields,  $a$  and  $\tilde{a}$  are dynamical gauge fields whose charge-1 fields are fermions (they are formally known as  $\text{spin}_c$  connections). We have also included *background*  $U(1)$  gauge fields  $B$  and  $B'$ .<sup>8</sup> Various background Chern-Simons terms are included to ensure that all the theories have the same response. Despite the profusion of background terms, the dynamical content of the theories in Eq. 11 is simple. Sec. III will also discuss various possibilities for the IR fates of the theories in the duality web, paying careful attention to symmetries.

There is a similar web of dualities for the  $SU(2)$ -invariant  $NCCP^1$  model, which we discuss in Sec. IV. The structure of this duality web is very similar to that of the easy-plane case: the  $SU(2)$ -invariant  $NCCP^1$  model is dual to the QED<sub>3</sub>-Gross-Neveu (QED-GN) theory with  $N_f = 2$ , and the two theories both admit their own self-dualities. We summarize the mutual

<sup>8</sup> Strictly speaking, we will take  $B+B'$  and  $B-B'$  to be properly quantized  $U(1)$  gauge fields.

dualities of the four theories in the web below:

$$\begin{aligned}
& \sum_{\alpha=1,2} |D_b z_\alpha|^2 - (|z_1|^2 + |z_2|^2)^2 \\
\iff & \sum_{\alpha=1,2} |D_{\bar{b}} w_\alpha|^2 - (|w_1|^2 + |w_2|^2)^2 \\
\iff & \sum_{j=1,2} \bar{\psi}_j i \not{D}_a \psi_j + \phi \sum_{j=1,2} \bar{\psi}_j \psi_j + V(\phi) \\
\iff & \sum_{j=1,2} \bar{\chi}_j i \not{D}_{\bar{a}} \chi_j - \phi \sum_{j=1,2} \bar{\chi}_j \chi_j + V(\phi),
\end{aligned} \tag{12}$$

where the potential of Ising scalar field  $V(\phi)$  is tuned to the critical point.

2. Understanding the duality web allows some powerful statements about emergent symmetries of possible IR fixed points for the two  $NCCP^1$  models which is one of the main goals of this paper. In the context of deconfined quantum criticality we will show that in both the easy plane and  $SU(2)$  symmetric cases the emergent symmetry enables rotating the Landau order parameters of the two phases on either side of the transition.

In the easy plane case, the duality web in its strongest form implies an emergent  $O(4)$  symmetry. The most basic local (gauge-invariant) operators are the order parameters  $(n_1, n_2, n_3, n_4)$ , which form a vector representation of the  $O(4)$  symmetry. Since  $SO(4) \sim (SU(2) \times SU(2))/\mathbb{Z}_2$ , the vector operators can be rearranged into  $SU(2)$  spinors. The complex doublet

$$(\Phi_1, \Phi_2) \sim (\overbrace{n_3 + in_4}^{\text{Néel}}, \overbrace{n_1 + in_2}^{\text{VBS}}) \tag{13}$$

is a fundamental under the first  $SU(2)$ , and  $(\Phi_1^*, -\Phi_2)$  is a fundamental under the second. The improper  $O(4)$  reflection is represented as complex conjugation on one of the components of  $\Phi$ . The two complex operators are represented in each theory as:

$$\begin{aligned}
(\Phi_1, \Phi_2) & \sim (z_1^\dagger z_2, \mathcal{M}_b) \\
& \sim (\mathcal{M}_{\bar{b}}^\dagger, w_2^\dagger w_1) \\
& \sim (\psi_2^\dagger \mathcal{M}_a, (\psi_1^\dagger \mathcal{M}_a)^\dagger) \\
& \sim \left( (\chi_2^\dagger \mathcal{M}_{\bar{a}})^\dagger, (\chi_1^\dagger \mathcal{M}_{\bar{a}})^\dagger \right),
\end{aligned} \tag{14}$$

where  $\mathcal{M}_b$  is the monopole (instanton) operator for the gauge field  $b$ . (In QED the monopole configuration induces one zero mode for each Dirac fermion, and gauge invariance requires

exactly one of the two zero modes to be filled [55]. The notation  $\psi_1^\dagger \mathcal{M}_a$  denotes a monopole in  $a$  with the zero mode from the Dirac fermion  $\psi_1$  filled.) The  $O(4)$  symmetry is discussed in detail in Sec. III. Its implications for numerical simulations are discussed in Sec. VIII.

In the  $SU(2)$ -invariant case, the duality web implies an emergent  $SO(5)$  symmetry. The most basic local operators form a vector representation of this  $SO(5)$ :  $(n_1, n_2, n_3, n_4, n_5)$ . They are represented in  $NCCP^1$  and QED-GN theories as

$$\begin{aligned}
(n_1, n_2, n_3, n_4, n_5) &\sim (2 \operatorname{Re} \mathcal{M}_b, 2 \operatorname{Im} \mathcal{M}_b, z^\dagger \sigma_x z, z^\dagger \sigma_y z, z^\dagger \sigma_z z) \\
&\sim (w^\dagger \sigma_x w, -w^\dagger \sigma_y w, 2 \operatorname{Re} \mathcal{M}_{\tilde{b}}, -2 \operatorname{Im} \mathcal{M}_{\tilde{b}}, w^\dagger \sigma_z w) \\
&\sim (\operatorname{Re}(\psi_1^\dagger \mathcal{M}_a), -\operatorname{Im}(\psi_1^\dagger \mathcal{M}_a), \operatorname{Re}(\psi_2^\dagger \mathcal{M}_a), \operatorname{Im}(\psi_2^\dagger \mathcal{M}_a), \phi) \\
&\sim (\operatorname{Re}(\chi_1^\dagger \mathcal{M}_{\tilde{a}}), -\operatorname{Im}(\chi_1^\dagger \mathcal{M}_{\tilde{a}}), \operatorname{Re}(\chi_2^\dagger \mathcal{M}_{\tilde{a}}), -\operatorname{Im}(\chi_2^\dagger \mathcal{M}_{\tilde{a}}), \phi) \quad (15)
\end{aligned}$$

This  $SO(5)$  symmetry is discussed in detail in Sec. IV. It has been numerically observed in Ref. [15], providing a strong support to our duality web. Its further implications are discussed in Sec. IV E and VIII.

3. The easy-plane theory has several  $\mathbb{Z}_2$  (or  $\mathbb{Z}_2$ -like) symmetries which are anomalous. In the context of lattice quantum magnetism these symmetries include the  $\mathbb{Z}_2$  spin-flip, time-reversal and lattice translation symmetries (see Sec. III B 2). Under these symmetries the Lagrangian is invariant only up to a background term:

$$\mathcal{L} \rightarrow \mathcal{L} - \frac{1}{2\pi} B_1 dB_2, \quad (16)$$

where  $B_1 = B - B'$  and  $B_2 = B + B'$  are the properly quantized background  $U(1)$  gauge fields. On the lattice this anomaly is harmless since one of the  $U(1)$  symmetries is really a discrete lattice rotation symmetry. However, if one wants to formulate the theory with all these symmetries realized in an on-site manner, the theory can only exist on the surface of a three-dimensional bulk. The symmetry anomaly can be canceled by a bulk mutual  $\Theta$ -term

$$- \frac{1}{4\pi} \int_{Bulk} dB_1 \wedge dB_2. \quad (17)$$

All the dualities on the surface are then related to the electric-magnetic dualities in the bulk. Many symmetry actions that appear to be complicated on the surface (in certain pictures) become obvious once the bulk view is taken. We discuss this in Sec. V for the easy plane theory.



4. None of the field theories in the duality web, Eq. (12), possesses the full  $SO(5)$  symmetry explicitly — the symmetry is at best emergent in the IR. Further, just as in the easy-plane case, the  $SO(5)$  symmetry is anomalously realized. We also discuss two (renormalizable) field theories with explicit  $SO(5)$  symmetry in Sec. VI. The first one is  $QCD_3$  with  $N_f = 2$ :

$$\mathcal{L} = \sum_{v=1,2} i\bar{\psi}_v \gamma^\mu (\partial_\mu - ia_\mu) \psi_v, \quad (18)$$

where  $a$  is an  $SU(2)$  gauge field, and  $\psi_{1,2}$  are two  $SU(2)$ -fundamental fermions. This theory can be obtained from the square lattice spin-1/2 model through a standard parton construction with a  $\pi$ -flux mean field ansatz, and it has an  $SO(5)$  symmetry which becomes manifest when (18) is written in terms of Majorana fermions. The second theory is a Higgs descendent of  $QCD_3$ , where the  $SU(2)$  gauge symmetry is Higgsed down to  $U(1)$ :

$$\mathcal{L} = \sum_{i=1}^4 i\bar{\psi}_i \gamma^\mu (\partial_\mu - ia_\mu) \psi_i + (\lambda \mathcal{M}_a + h.c.), \quad (19)$$

where  $a_\mu$  is now a  $U(1)$  gauge field, and the term  $\mathcal{M}_a$  represents (schematically) monopole tunneling (instanton) events. In both theories the Dirac fermions transform in the spinor representation of  $SO(5)$ . The  $SO(5)$ -vector operators are simply the mass operators that are time-reversal invariant.

While the IR fates of the theories (18) and (19) are unknown, both theories have the same symmetry anomaly as the deconfined critical point. Therefore, one possible scenario, among others, is that one of these theories will flow to the deconfined critical point in the IR.

5. The full  $SO(5)$  symmetry of the deconfined critical point is anomalous, as revealed already by the sigma model approach. The manifestly  $SO(5)$ -invariant QCD theory makes it possible to analyze the anomaly in more detail. We show in Secs. VIC and VII that  $QCD_3$  with  $N_f = 2$ , with the full  $SO(5) \times Z_2^T$  symmetry, can only be realized on the surface of a three-dimensional bosonic symmetry-protected topological (SPT) state. To characterize this SPT state, we can introduce a background  $SO(5)$  gauge bundle  $A^5$  in the  $3+1D$  bulk. The topological response to  $A^5$  is given by a discrete theta-angle (in contrast to the more familiar theta-angle in  $3+1D$  which can be continuously varied). Sec. VIC provides a physical derivation of these results which are then rederived by more formal methods in Sec. VII.<sup>9</sup> We show that the partition

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<sup>9</sup> A brief review of some math concepts relevant to this Sec. VII is given in Appendix D.

function of the 3 + 1D SPT for a given  $SO(5)$  gauge field configuration is

$$Z[A^5] = |Z[A^5]|e^{i\pi \int w_4[A^5]}, \quad (20)$$

where  $w_4[A^5]$  is known as the 4th Stiefel-Whitney class of the  $SO(5)$  gauge bundle. Though the IR fate of  $QCD_3$  with  $SO(5) \times Z_2^T$  symmetry is not known, we show that it must either break this symmetry spontaneously, or flow to a CFT. Gapped symmetry preserving phases (even with topological order) are prohibited.

6. We also discuss the implications of these dualities (for example, for numerical simulations) extensively in Sec. VIII. We outline a variety of numerical tests of many aspects of the dualities. We also show how numerical calculations on fermionic  $QED_3$  and  $QED_3$ -GN may provide a new handle on issues associated with deconfined critical points.
7. In Sec. IX we discuss what is currently known about the deconfined critical points from simulations. In particular we discuss the possibility of ‘pseudocritical’ behavior for deconfined critical points [14]. It is possible that the theories discussed in this paper do not flow to stable CFTs, which in the context of deconfined critical point means that the transition is ultimately first order. But the flow to instability could be very slow, giving rise to a very large correlation length, and scaling behavior can still hold up to this very large scale (with exponents drifting as the scale increases). This pseudocritical scenario could potentially reconcile various seemingly conflicting results from numerical simulations: the observance of the emergent  $SO(5)$  symmetry, the drifting of the scaling behavior, and the constraints on exponents of  $SO(5)$  invariant CFTs from conformal bootstrap [24, 25]. In this scenario, the dualities and emergent symmetries discussed in this paper can still hold below the (very large) correlation length.

We should point out that the pseudocritical scenario may be more broadly relevant to many quantum phase transitions: in such a scenario the system shows quantum critical behavior above a very low temperature scale  $T^*$ , below which the criticality eventually disappears.

Related field theoretic work on these dualities has recently appeared in Ref. [56] while this paper was in preparation.

### III. EASY PLANE $NCCP^1$ AND FERMIONIC $N_f = 2$ QED<sub>3</sub>

We propose a duality of the easy plane  $NCCP^1$  model to fermionic  $N_f = 2$  QED<sub>3</sub>:

$$|D_{b+B}z_1|^2 + |D_{b+B'}z_2|^2 - |z_1|^4 - |z_2|^4 - \frac{1}{2\pi}bd(B+B') - \frac{1}{2\pi}BdB' - \frac{1}{2\pi}B'dB' \quad (21)$$

$$\Longleftrightarrow \bar{\psi}_1 i \not{D}_{a-B} \psi_1 + \bar{\psi}_2 i \not{D}_{a+B} \psi_2 + \frac{1}{2\pi}adB' + \frac{1}{4\pi}(BdB - B'dB'). \quad (22)$$

Here  $b$  and  $a$  are the dynamical  $U(1)$  gauge fields,<sup>10</sup> and we have also included *background*  $U(1)$  gauge fields  $B$  and  $B'$ . Various background Chern-Simons terms are included to ensure that the theories on the two sides have the same response (we will elaborate on this below). In Appendix A, we present the above duality in a more precise and compact, but less physically intuitive, form. The identification of the Néel and VBS order parameters in the  $z$ -theory was reviewed in Sec. II A above. We will identify these order parameters in the fermionic description in Sec. III A below.

Both theories have been examined in independent numerical studies. This duality implies that two topical issues (fixed points of QED<sub>3</sub> at  $N_f = 2$  and the easy-plane  $NCCP^1$ ) are closely related. A ‘weak’ form of the duality is the assertion that easy-plane  $NCCP^1$  is equivalent to QED<sub>3</sub> perturbed by interactions which break the flavor  $SU(2)$  symmetry to  $U(1)$ . These perturbations are formally irrelevant at the free UV fixed point of QED<sub>3</sub> where the gauge field  $a_\mu$  is decoupled from the fermions. If these interactions are not important for the IR behavior, as one would naively guess, then the IR fate of easy plane  $NCCP^1$  will be the same as the fate of QED<sub>3</sub> with full flavor  $SU(2)$  symmetry, and the IR behavior will have an enlarged emergent symmetry if QED<sub>3</sub> flows to a nontrivial fixed point.

In the deconfined criticality context, the easy-plane models that have been studied show first order transitions [20, 57–60]. However, on the fermionic side, a recent numerical study [43] found evidence for an IR CFT in Eq. (8) with  $SU(2)$  flavor symmetry (in contrast to earlier calculations [46]). Recent numerics on the quantum phase transition between a particular 2D bosonic Symmetry Protected Topological (SPT) state and the trivial state, which presumably is described by Eq. (8) with tuning parameter  $\sum_{j=1}^2 m\bar{\psi}_j\psi_j$ , also show evidence of a continuous phase transition [44, 45].<sup>11</sup> In light of the duality it may be interesting to revisit easy plane deconfined criticality.

<sup>10</sup> Strictly speaking  $a$  is a  $\text{spin}_c$  connection.

<sup>11</sup> In Ref. [44, 45] the quantum phase transition was discussed with the  $O(4)$  nonlinear sigma model with a topological term, which according to Ref. [27] is the low energy effective field theory of the  $N_f = 2$  QED<sub>3</sub>.

### A. Justification for the duality

We first show how the fermionic dual of the easy plane  $NCCP^1$  model may be justified. This duality has already been mentioned in Ref. [33].

First recall the ‘basic’ boson-fermion duality [32] relating the free Dirac fermion to a Wilson-Fisher boson coupled to  $U(1)_1$ . There are two closely related versions of this duality. The Dirac side is (with our conventions)

$$\mathcal{L}_{f1} = i\bar{\psi}\not{D}_A\psi. \quad (23)$$

One version of the dual boson theory is

$$\mathcal{L}_{b1} = |D_b\phi|^2 - |\phi|^4 + \frac{1}{2\pi}bdA + \frac{1}{4\pi}bdb + \frac{1}{8\pi}AdA. \quad (24)$$

The other dual boson theory is

$$\mathcal{L}_{b2} = |D_{\hat{b}}\hat{\phi}|^2 - |\hat{\phi}|^4 - \frac{1}{2\pi}\hat{b}dA - \frac{1}{4\pi}\hat{b}d\hat{b} - \frac{1}{8\pi}AdA. \quad (25)$$

The two boson theories are simply related:  $\hat{\phi}$  is the vortex dual of  $\phi$ . Let us also recall the mapping of the relevant ‘mass’ operators. On the Dirac side, a mass term  $m\bar{\psi}\psi$  with  $m > 0$  maps to  $r|\phi|^2$  with  $r > 0$  in Eq. (24) while it maps to  $-\hat{r}|\hat{\phi}|^2$  with  $\hat{r} > 0$  in Eq. (25). This change of sign of the boson mass between the two bosonic theories is exactly what we expect given that  $\hat{\phi}$  is the vortex dual of  $\phi$ .

Now, starting with the interacting fermionic theory in Eq. (22), we use the dual theory in Eq. (24) for the first flavor of fermion and the dual theory in Eq. (25) for the second flavor. The resulting dual theory, in terms of bosons  $\phi_I$  and dynamical gauge fields  $b_I$  ( $I = 1, 2$ ) is

$$\begin{aligned} \mathcal{L}'_{b12} = \sum_{I=1,2} \mathcal{L}[\phi_I, b_I] &+ \frac{1}{2\pi}b_1d(a-B) + \frac{1}{4\pi}b_1db_1 + \frac{1}{8\pi}(a-B)d(a-B) - \frac{1}{2\pi}b_2d(a+B) \\ &- \frac{1}{4\pi}b_2db_2 - \frac{1}{8\pi}(a+B)d(a+B) + \frac{1}{2\pi}adB' + \frac{1}{4\pi}(BdB - B'dB'), \end{aligned} \quad (26)$$

where  $\mathcal{L}[\phi_I, b_I]$  contains the kinetic and potential terms for  $\phi_I$  and  $b_I$ . Integrating out the dynamical gauge field  $a$  will impose the following constraint:

$$b_1 - b_2 - B + B' = 0. \quad (27)$$

This implies that we can define a new dynamical gauge field  $b$  such that  $b_1 = b + B$ , and  $b_2 = b + B'$ . Then Eq. (26) becomes exactly the first line of Eq. (22) (after identifying  $\phi$  with  $z_1$  and  $\hat{\phi}$  with  $z_2$ ).

In addition to the formal derivation above, in the following we perform various consistency checks of the duality and make a few further comments. We will defer until Sec. III B a detailed discussion of the matching of symmetries (explicit or emergent) on the two sides, and their implications.

1. Clearly both sides have (at least)  $U(1) \times U(1)$  symmetry, probed by the two background gauge fields  $B$  and  $B'$ . On the boson side the gauge-invariant operator  $z_1^* z_2$  has charges  $q_B = +1$ ,  $q_{B'} = -1$ , and the monopole operator  $\mathcal{M}_b$  has  $q_B = q_{B'} = +1$ . It is actually more natural to define  $B_1 = B - B'$  and  $B_2 = B + B'$  and to define  $\Phi_{B_1}$  and  $\Phi_{B_2}$  as the order parameters charged under the corresponding global  $U(1)$ s. That is, if we let  $(q_1, q_2)$  denote the charges under  $U_{B_1}(1)$  and  $U_{B_2}(1)$ , then  $\Phi_{B_1}$  carries charges  $(1, 0)$  and  $\Phi_{B_2}$  carries  $(0, 1)$ . We have

$$\Phi_{B_1} = z_1^* z_2, \quad \Phi_{B_2} = \mathcal{M}_b. \quad (28)$$

These are the two Landau order parameters (Néel and VBS respectively in the quantum magnet realization), one of which orders on each side of the putative deconfined QCP.

On the fermion side, a monopole operator  $\mathcal{M}_a$  is associated with two complex fermion zero modes  $f_{1,2}$  from the two Dirac fermions, and gauge invariance requires filling one of the zero modes [55]. Therefore the operators  $f_j^\dagger \mathcal{M}_a$  are gauge invariant bosons with charges  $q_B = -1$ ,  $q_{B'} = -1$  (for  $j = 1$ ) and  $q_B = +1$ ,  $q_{B'} = -1$  (for  $j = 2$ ). Clearly they can be identified with the corresponding Landau order parameters on the bosonic side.

The bosonic side of Eq. (22), after a redefinition  $b \rightarrow b - B'$ , can be written as simply

$$|D_{b+B_1} z_1|^2 + |D_b z_2|^2 - |z_1|^4 - |z_2|^4 - \frac{1}{2\pi} b d B_2. \quad (29)$$

Below we will use both forms of the easy-plane  $NCCP^1$  Lagrangian.

2. The derivation above makes clear that the correspondence for mass operators is

$$m_1 \bar{\psi}_1 \psi_1 + m_2 \bar{\psi}_2 \psi_2 \quad \Longleftrightarrow \quad m_1 |z_1|^2 - m_2 |z_2|^2. \quad (30)$$

This is because we used the first boson-fermion duality (24) for the first fermion, and the second boson-fermion duality (25) for the second fermion. It is easy to check that the phases obtained by adding such mass operators match in the bosonic and fermionic descriptions:

On the bosonic side turning on  $r(|z_1|^2 + |z_2|^2)$  will drive the system into an ordered phase with  $\langle \Phi_{B_1} \rangle \neq 0$  or one with  $\langle \Phi_{B_2} \rangle \neq 0$  depending on the sign of  $r$ .<sup>12</sup> If  $r > 0$  we have

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<sup>12</sup> In the quantum magnetism realization these are simply the Néel and VBS ordered states.

$\langle z_1 \rangle, \langle z_2 \rangle \neq 0$ , a Higgsed gauge field  $b_\mu$ , and a nonzero expectation value  $\langle z_1^* z_2 \rangle \neq 0$  which leads to a ‘Meissner effect’ for  $B_1$ . If  $r < 0$  we have a free dynamical Maxwell photon at low energy which leads to the Meissner effect for  $B_2$ , via the mutual Chern Simons term in (29). The mass term corresponds on the fermionic side to  $r\bar{\psi}\sigma^z\psi$ , which gaps out the fermions. Integrating them out produces the term  $\frac{1}{2\pi}\text{sgn}(m)adB$ . Together with the  $\frac{1}{2\pi}adB'$  term in (22), this again leads to the Meissner effect of either  $B_1$  or  $B_2$  depending on the sign of  $m$ . This is consistent with the operator identification discussed above.

3. We can also turn on an anti-symmetric mass  $\mu(|z_1|^2 - |z_2|^2)$  on the bosonic side. For  $\mu > 0$  we have  $\langle z_1 \rangle \neq 0$  but  $\langle z_2 \rangle = 0$ , which gaps out  $b$  and sets  $b = -B$ . So we get a gapped phase with response  $\frac{1}{2\pi}(BdB - B'dB') = \frac{1}{2\pi}B_1dB_2$ . If we reinterpret  $B$  as a ‘charge’ probe and  $B'$  as a ‘spin’ probe, this corresponds to the response of the bosonic integer quantum hall state [61, 62]. For  $\mu < 0$  we get a gapped phase with a trivial response. (In the context of deconfined criticality in quantum magnets, where  $U(1)_{B_2}$  is only emergent, both of these are trivial phases in which the Néel order parameter is polarized along the  $z$  axis.)

On the fermionic side these phases can be reproduced by a symmetric mass term  $\mu(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2)$ : integrating out the fermions leaves the action

$$\frac{1}{4\pi}\text{sgn}(\mu)(ada + BdB) + \frac{1}{2\pi}adB' + \frac{1}{4\pi}(BdB - B'dB'). \quad (31)$$

Since the term  $\pm\frac{1}{4\pi}ada$  is a trivial topological field theory, it can be integrated out safely and we produce the same response theory as on the bosonic side. It is also easy to see that no thermal hall conductance (i.e. no chiral edge state) is generated for either phase (in either picture), in agreement with known results [61, 62] for the trivial insulator and the bosonic integer quantum hall state.

The responses above indicate that the massless theories of primary interest to us can also be viewed as describing the phase transition between the trivial and bosonic integer quantum Hall insulators. The fermionic picture for this transition was discussed in Refs. [63, 64] using parton constructions (see also Appendix B for a different perspective) and more recently in Ref. [65] using a coupled-wire construction. This bosonic IQH transition has been seen numerically, and the results show evidence of a continuous phase transition [44, 45].

The mean field phase diagram implied by the above is summarized in Fig. 1, both in the deconfined criticality context (where  $U(1)_{\text{Néel}}$  is an exact microscopic spin symmetry and

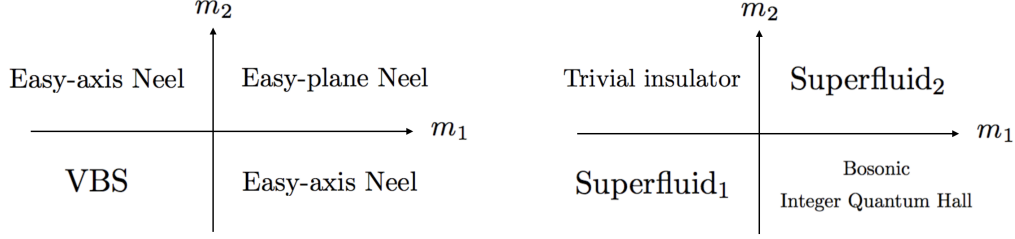


FIG. 1: Mean-field phase diagram for the mass term  $m_1|z_1|^2 + m_2|z_2|^2$  in the easy-plane  $NCCP^1$  model. The same phase diagram is obtained from the QED theory with mass term  $m_1\bar{\psi}_1\psi_1 - m_2\bar{\psi}_2\psi_2$ . The left panel is realized in the context of the quantum magnet, where the  $U(1)_{B_2}$  symmetry is only emergent. The right panel is realized in the context of the integer quantum hall transition of bosons, where the two superfluid phases correspond to superfluids of the first or second layer, i.e. up or down components of the pseudospin. (Going beyond mean field will move the phase boundaries away from the axes.)

$U(1)_{\text{VBS}}$  is emergent at the critical point) and for the boson integer quantum Hall transition. This mean field picture captures only the topology of the phase diagram adjacent to the putative critical point: in general the phase boundaries will meet at a cusp (since  $|z_1|^2 + |z_2|^2$  and  $|z_1|^2 - |z_2|^2$  will have different scaling dimensions) and will not lie along the axes.

4. There is potentially a dual formulation of  $N = 2$  QED<sub>3</sub> directly in terms of the bosonic monopoles  $f_j^\dagger \mathcal{M}_a$  which are precisely the  $\Phi_{B_1}, \Phi_{B_2}$  defined in Eq. (28). The most we know about this theory if we want to formulate it directly in terms of the physical bosons is that it has the structure of an (anisotropic)  $O(4)$  sigma model at  $\theta = \pi$  [27] discussed in Sec. II A. This is consistent with the various other known connections between this sigma model and the QED<sub>3</sub> theory. This effective field theory is a convenient language for discussing the emergent  $O(4)$  symmetry that would be required if the strong duality conjecture holds for these theories.

## B. Symmetries

We now study the symmetries of the various dual actions, and the implications of the dualities for emergent symmetries of the easy-plane deconfined quantum critical points. Here we will discuss the symmetries from a physical point of view natural in condensed matter physics. This will make

plausible the statements about non-trivial emergent symmetries.

### 1. Continuous symmetries

In this section we will see how the duality web leads to the possibility that the easy plane  $NCCP^1$  theory could have an emergent  $O(4) \times Z_2^T$  symmetry.<sup>13</sup> In Sec. III C we will express the requirements for this symmetry enhancement more formally in terms of the properties of the putative ‘mother’  $O(4)$  fixed point. Here we discuss how the dualities, if they hold in a ‘strong’ form, lead to this emergent symmetry.

In the duality between easy-plane  $NCCP^1$  and  $QED_3$  we naively only expect that the continuous symmetry of the resulting fermion theory is  $U(1) \times U(1)$ . As explained in Sec. II B, the fermionic action we have written down apparently however has manifest  $\frac{SU(2) \times U(1)}{Z_2}$  symmetry with the  $SU(2)$  corresponding to rotations between the two fermion flavors. Thus, we should allow for terms that break this apparent flavor  $SU(2)$  down to  $U(1)$ . The mass term  $\bar{\psi}\sigma^z\psi$  will accomplish that but this is precisely the operator whose coefficient is tuned to zero at the transition. The minimal operators with no derivatives that break the flavor  $SU(2)$  symmetry are thus four fermion terms, *e.g.*  $(\bar{\psi}\sigma^z\psi)^2$ . At the free Dirac fermion fixed point these operators are strongly irrelevant. So if the  $SU(2)$  symmetric QED theory has a nontrivial IR fixed point, it is plausible that perturbations breaking  $SU(2)$  to  $U(1)$  are also irrelevant here, and that the theory with microscopic  $U(1)$  flavor symmetry flows to this point and has emergent  $SU(2)$  flavor symmetry in addition to the other symmetries already present.

Now consider the self-duality of  $QED_3$ . In the derivation of Ref. [31] of this self-duality a priori we only know that the  $U(1) \times U(1)$  symmetry of one side maps to the  $U(1) \times U(1)$  symmetry of the other side, with the role of the two  $U(1)$  symmetries being exchanged by duality (the flavor conservation symmetry on one side becomes the flux conservation symmetry on the other side). Again it is naively plausible that the  $SU(2)$  flavor symmetry emerges in the infrared on both sides of the duality (see Sec. III C for a more careful discussion). The two flavor  $SU(2)$  symmetries on the two sides are distinct from each other, implying that the full continuous symmetry of the  $QED_3$  theory is then  $(SU(2) \times SU(2))/Z_2 = SO(4)$ . A version of this argument was previously made in Ref. [35, 66]<sup>14</sup>. We will clarify the conditions under which this symmetry enhancement actually

<sup>13</sup> Here  $Z_2^T$  refers to time reversal which is antiunitary.

<sup>14</sup> To be completely precise Ref. [35] stated that the enhanced symmetry is  $Spin(4)$  while according to our discussion



happens.

On the fermionic side, the two flavors of Dirac fermions  $\psi_j$  form a spin-1/2 representation under one of the two  $SU(2)$  subgroups in  $SO(4)$ , and the dual Dirac fermions  $\chi_j$  form a spin-1/2 representation under the other  $SU(2)$  subgroup. These fermions do not transform in a simple way under the whole  $SO(4)$  group, but this is not problematic since they are not gauge invariant. The gauge invariant operators  $\mathcal{M}_a^\dagger f_j$  transform as a spinor under the flavor  $SU(2)$  of the  $\psi$  theory. As these are identified with the boson operators  $z_1^* z_2$  and  $\mathcal{M}_b$  it follows that these two operators are a spinor under this  $SU(2)$ . Thus this  $SU(2)$  rotates

$$\begin{pmatrix} \Phi_{B_1} \\ \Phi_{B_2} \end{pmatrix} \sim \begin{pmatrix} n_3 + in_4 \\ n_1 + in_2 \end{pmatrix} \quad (32)$$

as a spinor. It is easy to see that under the flavor  $SU(2)$  of the dual QED<sub>3</sub> theory,

$$\begin{pmatrix} \Phi_{B_1}^* \\ -\Phi_{B_2} \end{pmatrix} \sim \begin{pmatrix} n_3 - in_4 \\ -n_1 - in_2 \end{pmatrix} \quad (33)$$

is rotated as a spinor. This means that the  $SO(4)$  simply rotates the four real components of  $\Phi_{B_1}, \Phi_{B_2}$ , i.e.  $(n_1, n_2, n_3, n_4)$ , into one another. In the quantum magnetism realization these are precisely the Néel and VBS order parameters.

## 2. Discrete symmetries

Let us now turn to discrete symmetries. We have already mentioned the  $Z_2$  spin flip symmetry  $\mathcal{S}$ . For the quantum magnetism realization in spin-1/2 square lattice magnets, we must also discuss lattice translation, lattice rotation, lattice reflection, and time reversal symmetries.

### 1. $Z_2$ spin flip symmetry

The  $Z_2$  spin flip  $\mathcal{S}$  corresponds in the microscopic spin model to a rotation of the spin at each site by  $\pi$  around the  $x$ -axis. This is a subgroup of spin  $SO(3)$  symmetry that is presumed to be retained in the easy plane model. In the context of easy-plane deconfined criticality, this symmetry ensures that the only tuning parameter across the transition is the *symmetric* mass term  $r(|z_1|^2 + |z_2|^2)$ .

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it is  $SO(4)$ . In particular we will not find local operators transforming under the  $(\frac{1}{2}, 0)$  or  $(0, \frac{1}{2})$  representations.

The full action of  $\mathcal{S}$  in the easy-plane  $NCCP^1$  theory in the presence of background fields is:

$$z_1 \leftrightarrow z_2, \quad B \leftrightarrow B', \quad b \rightarrow b. \quad (34)$$

Equivalently,  $\mathcal{S}$  takes  $B_1 \leftrightarrow -B_1$ ,  $B_2 \leftrightarrow B_2$ . As emphasized in Sec. II A, the corresponding action on the Néel and VBS order parameters is

$$\mathcal{S}(\Phi_{B_1}) = \Phi_{B_1}^*, \quad \mathcal{S}(\Phi_{B_2}) = \Phi_{B_2}. \quad (35)$$

Thus this  $Z_2$  acts as an improper  $O(4)$  rotation on the vector  $(n_1, n_2, n_3, n_4)$  formed from the four real components of these fields.

In QED<sub>3</sub>,  $\mathcal{S}$  is a transformation between  $\psi_j$  and its dual fermion  $\chi_j$ . Since the mass terms transform under the fermion self-duality as  $\bar{\psi}_1\psi_1 \leftrightarrow -\bar{\chi}_2\chi_2$ , the antisymmetric fermion mass term  $\bar{\psi}\sigma^z\psi$  is invariant, which is consistent with the operator identification described above.

We should point out that in the continuum field theory this symmetry is actually anomalous. In both the boson and fermion pictures the Lagrangian picks up an extra term under this symmetry operation:

$$\mathcal{L} \longrightarrow \mathcal{L} + \frac{1}{2\pi}(B'dB' - BdB) = \mathcal{L} - \frac{1}{2\pi}B_1dB_2. \quad (36)$$

For deconfined criticality realized in a lattice spin system, this anomaly is harmless because the  $U(1)$  symmetry probed by  $B_2$  is really a discrete lattice rotation symmetry. However, if the symmetries are on-site, this theory can only be regularized on the surface of a three dimensional bulk. We shall discuss this in more detail in Sec. V.

Including time reversal which we discuss below, the full symmetry of the easy plane  $NCCP^1$  fixed point may thus be  $O(4) \times Z_2^T$ . Note that the enlargement of  $SO(4)$  to  $O(4)$  is also expected from the standpoint of the non-linear sigma model with a theta term, Eq. (7). If  $O(4)$  is broken to  $SO(4)$  the value of  $\theta$  can be varied away from  $\pi$ : this is plausibly a relevant perturbation.

## 2. Bosonic self-duality symmetry

There is also a  $Z_2$  subgroup, which we denote  $\mathcal{S}_\psi$ , of the  $SU(2)$  flavor symmetry of the QED<sub>3</sub>:

$$\mathcal{S}_\psi : \psi_1 \leftrightarrow \psi_2, \quad B \rightarrow -B, \quad a \rightarrow a, \quad B' \rightarrow B'. \quad (37)$$

This  $\mathcal{S}_\psi$  symmetry is not a microscopic symmetry for the quantum magnet. In  $NCCP^1$  it becomes the bosonic self-duality,  $z_\alpha \rightarrow w_\alpha$ :

$$\Phi_{B_1} \leftrightarrow \Phi_{B_2}^*. \quad (38)$$

This also shows that the  $SU(2)$  flavor group of the  $QED_3$  theory must act in a highly nonlocal fashion in the  $NCCP^1$  theory.

On the QED side, imposing  $\mathcal{S}_\psi$  symmetry forces the mass term to be symmetric,  $m\bar{\psi}_j\psi_j$ , which gives a transition between two distinct (SPT) gapped phases. On the bosonic side, this symmetry allows the antisymmetric mass term  $\mu(|z_1|^2 - |z_2|^2)$ , which also gives the SPT transition.

### 3. Time reversal

We now specialize to realizations of these deconfined critical points at the Néel-VBS transition in square lattice spin-1/2 magnets. Microscopic symmetries then include — in addition to the spin rotation and spin flip  $\mathcal{S}$  symmetries — time reversal and lattice symmetries.

Time reversal  $T$  is anti-unitary and acts on the  $NCCP^1$  fields as

$$T(z_\alpha) = \epsilon_{\alpha\beta} z_\beta \quad (39)$$

$$T(b) = -b - B - B' \quad (40)$$

$$T(B) = B, \quad T(B') = B'. \quad (41)$$

(For the gauge fields we only indicate the time reversal action on the spatial components; the time component will transform with the opposite sign.) Here  $\epsilon = i\sigma_y$  is antisymmetric with  $\epsilon_{12} = 1$ . This is consistent with  $\Phi_{B_1} \rightarrow -\Phi_{B_1}^*$  and  $\Phi_{B_2} \rightarrow \Phi_{B_2}^*$ , as befits the Néel and VBS order parameters respectively (written in complex form). Note that, as with the  $Z_2$  spin flip symmetry, the bosonic Lagrangian is only invariant up to an anomaly:

$$\mathcal{L} \rightarrow \mathcal{L} + \frac{1}{2\pi}(B'dB' - BdB) = \mathcal{L} - \frac{1}{2\pi}B_1dB_2. \quad (42)$$

On the dual  $QED_3$  side, time-reversal acts as the product  $\mathcal{S}_\psi\mathcal{T}$  transformation under which

$$T(\psi) = \mathcal{S}_\psi\gamma_0\psi, \quad T(a) = -a. \quad (43)$$

Note that the  $QED_3$  theory also has the same anomaly in Eq. (42).

#### 4. Translation symmetry

It suffices to discuss unit lattice translations along one direction, say the  $y$ -direction ( $x \rightarrow x$ ,  $y \rightarrow y + a$  where  $a$  is the lattice spacing) which we dub  $T_y$ . The Néel and VBS orders clearly transform as

$$T_y(\Phi_{B_1}) = -\Phi_{B_1} \qquad T_y(\Phi_{B_2}) = \Phi_{B_2}^*. \quad (44)$$

In the  $NCCP^1$  theory, this is implemented as

$$T_y(z_\alpha) = \epsilon_{\alpha\beta} z_\beta^*, \quad (45)$$

$$T_y(b) = -b \quad (46)$$

$$T_y(B) = -B', \quad T_y(B') = -B. \quad (47)$$

Just like with the  $Z_2$  spin flip, or time reversal symmetry, the  $NCCP^1$  Lagrangian is invariant under  $T_y$  only up to an anomaly (shift by  $\frac{1}{2\pi} B_1 dB_2$ ). On the QED<sub>3</sub> side,  $T_y$  takes the fermions  $\psi$  to their fermionic duals  $\chi$  though the detailed transformation is different from that of the  $Z_2$  spin flip symmetry. Specifically we have

$$\psi_1 \leftrightarrow -\chi_1^\dagger \qquad \psi_2 \leftrightarrow \chi_2^\dagger \qquad a \leftrightarrow -\tilde{a}. \quad (48)$$

Note that the symmetric mass  $\bar{\psi}\psi$  is odd under  $T_y$  while the anti-symmetric mass  $\bar{\psi}\sigma^z\psi$  is even, exactly as expected from the corresponding operators on the bosonic side.

A simple way to understand the action of  $T_y$  is as follows. From Eq. (44) we see that the action of  $T_y$  on the physical order parameters is similar to that of  $\mathcal{S}$  if we exchange the Néel and VBS orders. To make this precise consider the modified translation  $\tilde{T}_y = U_{B_1}(\pi)T_y$  which combines translation with a  $\pi$  rotation of the easy plane Néel vector. Then the physical order parameters transform as

$$\tilde{T}_y(\Phi_{B_1}) = \Phi_{B_1}, \qquad \tilde{T}_y(\Phi_{B_2}) = \Phi_{B_2}^*. \quad (49)$$

Thus we see that it is precisely the  $\mathcal{S}$  operation performed after a charge-conjugation transformation  $\mathcal{C}$  that takes all the fields to their charge-conjugate. This identification is nicely consistent with the action of  $T_y$  in the QED<sub>3</sub> theory in Eq. (48). The extra  $(-)$  sign in Eq. (48) is simply due to the additional  $U_{B_1}(\pi)$  in the definition of  $\tilde{T}_y$ .

### 5. Rotation and reflection symmetries

Lattice rotations by  $\frac{\pi}{2}$  about a square lattice site act very simply on the Néel and VBS vectors.

We have

$$R_{\frac{\pi}{2}}(\Phi_{B_1}) = \Phi_{B_1}, \quad R_{\frac{\pi}{2}}(\Phi_{B_2}) = i\Phi_{B_2}. \quad (50)$$

This is part of a  $U_{B_2}(1)$  rotation whose role in the various dualities we have already discussed.

A site-centered lattice reflection  $R_y$ , say about the  $x$ -axis ( $x \rightarrow x, y \rightarrow -y$ ) acts as

$$R_y(\Phi_{B_1}) = \Phi_{B_1}, \quad R_y(\Phi_{B_2}) = \Phi_{B_2}^*. \quad (51)$$

To define its action simply let us denote, for any gauge field  $A = (A_0, A_x, A_y)$  the reflected version by  $RA = (A_0, A_x, -A_y)$ . Then  $R_y$  acts in the  $z$ -formulation as

$$R_y(z_\alpha) = z_\alpha, \quad R_y(b) = Rb + RB + RB', \quad (52)$$

$$R_y(B) = -RB', \quad R_y(B') = -RB. \quad (53)$$

It is readily checked that the  $NCCP^1$  Lagrangian is invariant under this transformation and there is no anomaly. On the fermion side,  $R_y$  again involves a duality transformation between  $\psi$  and  $\chi$ :

$$\psi_1 \rightarrow \gamma_y \chi_2, \quad \psi_2 \rightarrow \gamma_y \chi_1, \quad a \rightarrow R\tilde{a}. \quad (54)$$

### C. Allowed symmetry breaking terms

The strong forms of the dualities discussed here involve the emergence of higher symmetries than are present in the UV Lagrangians. In order for the dualities to hold in the IR without fine-tuning, the hypothetical higher-symmetry fixed point must exist and must be stable to all perturbations allowed by the symmetry of the UV theory. Here we clarify these stability requirements. As will be discussed in Sec. IX, it is also possible that there is no fixed point with the higher symmetry, but that there is a pseudocritical regime up to a large but finite lengthscale  $\xi$ ; in this case the requirements should be interpreted in terms of the effective scaling dimensions in this regime.

We consider perturbations of the hypothetical  $O(4)$ -symmetric point relevant to  $N_f = 2$  QED<sub>3</sub> and the easy-plane  $NCCP^1$  model. Here we will take QED<sub>3</sub> to be defined with full  $SU(2)$  flavor symmetry, as done for instance in the lattice calculations of Ref. [43]. We will see that the conditions for the emergence of  $O(4)$  are more stringent for easy-plane  $NCCP^1$  than for QED. Therefore in

principle it is possible that the self-duality of QED<sub>3</sub> could hold, with emergent O(4), but that easy-plane  $NCCP^1$  could fail to flow to this fixed point. (By contrast, the requirements for the emergence of SO(5) are similar for the bosonic and fermionic theories, as we will see in Sec. IV D.)

To begin, the hypothetical fixed point must be stable to O(4)-singlet scalar perturbations. We certainly expect a relevant perturbation which is invariant under  $SO(4) = [SU(2) \times SU(2)]/Z_2$  but not under improper O(4) transformations: in the language of the sigma model for the field  $(n_1, n_2, n_3, n_4)$  (Secs. II A, III B 1) this corresponds to varying the coefficient of the  $\theta$  term away from  $\pi$ . But this perturbation is harmless as it is forbidden by time reversal, and in the easy-plane  $NCCP^1$  model also by the  $Z_2$  spin-flip symmetry  $\mathcal{S}$  (Sec. III B 2).

Apart from the O(4) vector order parameters  $n_a$  defined in Eq. 32 it is natural to expect the next leading scalar operators to be those in the two and four-index symmetric tensor representations of O(4). We denote these  $X_{ab}^{(2)}$  and  $X_{abcd}^{(4)}$ . At the level of symmetry,

$$X_{ab}^{(2)} \sim n_a n_b - \delta_{ab} n^2/4, \quad X_{abcd}^{(4)} \sim n_a n_b n_c n_d - (\dots), \quad (55)$$

where the subtraction  $(\dots)$  makes the operator traceless.  $X^{(2)}$  is certainly relevant.

The two-index symmetric tensor  $X_{ab}^{(2)}$  corresponds to the  $(1, 1)$  representation of  $SU(2) \times SU(2)$ . All components of this operator are therefore forbidden by the explicit SU(2) of QED. In the easy plane model one component,  $\sum_{a=1}^2 X_{aa}^{(2)}$ , is allowed but is precisely the tuning parameter for the Néel-VBS transition: i.e.  $\frac{1}{2}(|\Phi_{B_2}|^2 - |\Phi_{B_1}|^2)$  in terms of the complex Néel and VBS order parameters. The four-index symmetric tensor  $X^{(4)}$  is the  $(2, 2)$  representation of  $SU(2) \times SU(2)$ , so again all components are forbidden by the explicit SU(2) of QED. However in the easy-plane  $NCCP^1$  model the Néel-VBS anisotropy  $\sum_{a=1}^2 \sum_{b=3}^4 X_{aabb}^{(4)}$  is allowed, and microscopic models on the square lattice will also allow  $\sum_{a=1}^2 X_{aaaa}^{(4)}$ .<sup>15</sup>

Since the easy plane model allows an O(4)-breaking perturbation that is forbidden for QED, it is conceivable that the QED self-duality holds, with emergent O(4), but that the strong duality with easy-plane  $NCCP^1$  does not hold. This scenario would apply if there was an O(4) (pseudo)critical regime in which  $X^{(4)}$  was relevant, but the perturbations allowed in the QED theories were irrelevant.

The explicit SU(2) of QED restricts such perturbations to representations of the form  $(0, \text{integer})$ . In the sigma model language, the simplest such terms involve two derivatives and four powers of

<sup>15</sup> In fact for the lattice antiferromagnet another symmetry-breaking term is allowed, which locks VBS rotations to spatial rotations:  $(\partial_x n_1)^2 + (\partial_y n_2)^2$ . See also the note in Sec. IV D. (This perturbation is absent for the easy-plane  $NCCP^1$  field theory, where the VBS  $U(1)$  is an exact internal symmetry.)

$n$ , so are plausibly irrelevant, as argued in Ref. [27]. In other words, if there is an  $O(4)$  fixed point which is stable to  $O(4)$ -singlet perturbations, it is very likely that QED flows to it.

From the point of view of the fermionic Lagrangians, both types of perturbation (those allowed in the easy-plane model and those allowed in QED) can be cast as four-fermion terms, giving a plausibility argument for their irrelevance. The breaking of the  $SU(2) \times U(1)$  of QED to the  $U(1) \times U(1)$  of the easy plane model is accomplished by a four-fermion operator, as discussed in Sec. IIIB 1. For the possible emergence of  $SU(2) \times SU(2)$ , consider the following ‘weak’ form of the fermion-fermion duality. We expect that in principle there is an exact UV duality between a cutoff version of QED with  $SU(2)_B \times U(1)_{B'}$  symmetry and an effective field theory version of QED (with a highly fine-tuned Lagrangian) in which the explicit symmetry is  $U(1)_B \times U(1)_{B'}$ . (For convenience we label the groups by the probe field for the  $U(1)$  part.) Since the simplest terms that break  $SU(2)_{B'}$  are four-fermion terms in this dual description, it is plausible that  $SU(2)_{B'}$  will emerge in the continuum, giving full  $SU(2)_B \times SU(2)_{B'}$ .

#### IV. $SU(2)$ SYMMETRIC $NCCP^1$ AND FERMIONIC QED<sub>3</sub>–GN

We now turn to the deconfined critical point with full  $SU(2)$  symmetry. We propose a duality between the  $SU(2)$  symmetric  $NCCP^1$  model and the  $N_f = 2$  QED<sub>3</sub>–GN theory,

$$\sum_{\alpha=1,2} |D_b z_\alpha|^2 - (|z_1|^2 + |z_2|^2)^2 \quad (56)$$

$$\Longleftrightarrow \sum_{j=1,2} \bar{\psi}_j i \not{D}_a \psi_j + \phi \sum_{j=1,2} \bar{\psi}_j \psi_j + V(\phi), \quad (57)$$

where  $\phi$  is a critical Ising field (real scalar), with the Ising terms  $(\partial\phi)^2 - \phi^4$  suppressed for notational convenience. The QED<sub>3</sub>–GN model has not been studied numerically as far as we know. The duality suggests the possibility of critical behavior with emergent symmetry.

To justify this duality, we consider the phases of the QED<sub>3</sub>–GN Lagrangian in Eq. (57), making use of the results for the pure QED in Sec III. First, the phase with a positive mass term for  $\phi$  and  $\langle\phi\rangle = 0$  is expected to be equivalent to QED<sub>3</sub>, and dual to the critical easy plane  $NCCP^1$  theory. What does the coupling to  $\phi$  mean in the bosonic theory? The mass term  $\bar{\psi}\psi$  is identified with  $(|z_1|^2 - |z_2|^2)$  in Eq. 30, so the coupling to the scalar field becomes

$$\phi (|z_1|^2 - |z_2|^2). \quad (58)$$

Now consider the QED<sub>3</sub>-GN theory in the phase where  $\langle\phi\rangle \neq 0$ , induced by turning on a negative mass term for  $\phi$ . This gives the symmetric mass term for QED<sub>3</sub> and the antisymmetric mass for easy-plane  $NCCP^1$ , as discussed under Eq. (31). The theory becomes trivially gapped (except for response terms). Finally, what does the phase transition associated with the onset of  $\langle\phi\rangle$  of the QED<sub>3</sub>-GN model correspond to on the boson side? We propose that it is the critical point of the  $SU(2)$  invariant  $NCCP^1$  model.

The discussion of the phase diagram above gives a basic consistency check on this duality. The operator identification goes as follows: the Ising field  $\phi$  on the fermionic side is dual to  $|z_1|^2 - |z_2|^2$  on the bosonic side. The Ising mass  $-\lambda\phi^2$  is dual to the anisotropy  $\lambda|z_1|^2|z_2|^2$  on the bosonic side.<sup>16</sup> The phases with  $\langle\phi\rangle = 0$  and  $\langle\phi\rangle \neq 0$  correspond respectively to the easy-plane critical theory and to a gapped state with easy-axis Néel order.

To be more precise, the duality with the critical  $SU(2)$  invariant  $NCCP^1$  model requires an emergent  $SO(5)$  symmetry: the basic assumption underlying the duality is that allowed terms in each theory which break this symmetry are irrelevant. We will make this explicit in Sec. IV D. Once again we postpone a detailed discussion of the matching of the symmetries of the two sides and their implications until Sec. IV C.

### A. Duality from the sigma model

We now provide an alternative understanding of the proposed duality, from the standpoint of the non-linear sigma model description of the  $SU(2)$  invariant  $NCCP^1$  model. We propose the equivalence between the 2+1D  $SO(5)$  nonlinear sigma model with a WZW term at level 1 (extended to a strong-coupling fixed point) and  $N = 2$  QED<sub>3</sub> deformed with a quartic interaction term to a critical point at  $\lambda = \lambda_c$ :

$$\begin{aligned} \mathcal{L} &= \frac{1}{g}(\partial_\mu \mathbf{n})^2 + \frac{2\pi}{\Omega_4} \int_0^1 du \, n^a \partial_u n^b \partial_x n^c \partial_y n^d \partial_t n^e \\ \iff \mathcal{L} &= \sum_{j=1}^2 \bar{\psi}_j i \not{D}_a \psi_j + \frac{\lambda}{2} \left( \sum_{j=1}^2 \bar{\psi}_j \psi_j \right)^2. \end{aligned} \quad (59)$$

<sup>16</sup> Strictly speaking the anisotropy term dual to the Ising mass should be  $\lambda[|z_1|^2|z_2|^2 - \alpha(|z_1|^2 + |z_2|^2)]$ , where the second term is added to keep the theory critical when  $\lambda > 0$ . We discuss this further in Sec. IV E.



We also identify a relevant perturbation on both sides of the duality:

$$u(n_5)^2 \sim u \left( \sum_{j=1}^2 \bar{\psi}_j \psi_j \right)^2. \quad (60)$$

When  $u > 0$  we expect that  $n_5$  can be treated as effectively zero in the  $\text{SO}(5)$  sigma model, so that this theory reduces to an  $\text{O}(4)$  nonlinear sigma model with a  $\Theta$ -term at  $\Theta = \pi$ :

$$\mathcal{L} = \frac{1}{g}(\partial_\mu \mathbf{n})^2 + \frac{\Theta}{\Omega_3} n^a \partial_t n^b \partial_x n^c \partial_y n^d, \quad \Theta = \pi. \quad (61)$$

This theory has been shown [27], with some assumptions, to be the low energy effective theory of  $N = 2$   $\text{QED}_3$ . Consistent with this, a positive  $u$  in the  $\text{QED}_3$ -Gross-Neveu model will drive the system back to  $\text{QED}_3$ .

When  $u < 0$ , the  $\text{SO}(5)$  sigma model develops a nonzero expectation value  $\langle n_5 \rangle$ . This spontaneously breaks the  $Z_2$  subgroup of the  $\text{SO}(5)$ . Depending on the sign of  $\langle n_5 \rangle$ , the effective  $\text{O}(4)$  sigma model for the remaining components (which we may imagine deriving by integrating out fluctuations in  $n_5$ ) will flow to either  $\Theta = 2\pi$  or  $\Theta = 0$ . The  $\text{QED}_3$ -Gross-Neveu model with  $u < 0$  spontaneously condenses  $\langle \sum_j \bar{\psi}_j \psi_j \rangle$ . This precisely yields the two phases with  $\Theta = 0$  and  $2\pi$ . A condensate of  $\langle \sum_j \bar{\psi}_j \psi_j \rangle$  spontaneously breaks the  $Z_2$  subgroup of  $\text{O}(4)$  (the  $Z_2$  transformation that exchanges the two  $\text{SU}(2)$  subgroups), which is also the  $Z_2$  subgroup of  $\text{SO}(5)$ :  $Z_2$  takes  $(n_1, n_2, n_3, n_4, n_5)$  to  $(-n_1, -n_2, -n_3, n_4, -n_5)$ .

## B. Self-dualities

We showed that the easy-plane duality Eq. (22) naturally motivates the duality of the  $\text{SU}(2)$  symmetric  $NCCP^1$  theory in Eq. (56). Following the same logic, the self-dualities of easy-plane  $NCCP^1$  and the  $\text{QED}_3$  theories also motivate further self-dualities with higher symmetries. For the  $NCCP^1$  model, this self-duality reads

$$\begin{aligned} & \sum_{\alpha=1,2} |D_b z_\alpha|^2 - (|z_1|^2 + |z_2|^2)^2 \\ \iff & \sum_{\alpha=1,2} |D_{\tilde{b}} w_\alpha|^2 - (|w_1|^2 + |w_2|^2)^2. \end{aligned} \quad (62)$$

In the dual theory, the  $U(1)$  phase rotation symmetry of the local operator  $w_1^* w_2$  corresponds to the flux conservation of the  $b$  gauge field in the original theory; likewise the flux conservation of  $\tilde{b}$

in the dual theory corresponds to the  $U(1)$  phase rotation symmetry of  $z_1^* z_2$  in the original theory. One can check the consistency of this duality by turning on an anisotropy term

$$\lambda |z_1|^2 |z_2|^2 \sim \lambda |w_1|^2 |w_2|^2. \quad (63)$$

When  $\lambda > 0$ , both theories flow to easy-plane  $NCCP^1$ , where the self-duality holds.<sup>17</sup> When  $\lambda < 0$ , both theories flow to the easy-axis limit, where the system becomes trivially gapped.

Similarly for the  $QED_3$ -GN model we have the self-duality

$$\begin{aligned} & \sum_{j=1,2} \bar{\psi}_j i \not{D}_a \psi_j + \phi \sum_{j=1,2} \bar{\psi}_j \psi_j + V(\phi) \\ \iff & \sum_{j=1,2} \bar{\chi}_j i \not{D}_{\tilde{a}} \chi_j - \phi \sum_{j=1,2} \bar{\chi}_j \chi_j + V(\phi). \end{aligned} \quad (64)$$

The switching of the two global  $U(1)$  symmetries in the two pictures is similar to that in the self-duality of  $NCCP^1$ . The consistency of this duality is checked by turning on a mass term for the Ising scalar fields on both sides: when  $\langle \phi \rangle \neq 0$  both sides are trivially gapped, and when  $\phi$  is gapped the duality follows from the self-duality of  $QED_3$ .

### C. Symmetries

We now study the symmetries of the  $NCCP^1$  theory and its duals. We simply assume the correctness of the proposed dualities.

The bosonic side has a manifest  $SO(3) \times U(1)$  symmetry, where the  $z$  bosons are  $SO(3)$  spinors, and the global  $U(1)$  symmetry is simply the flux symmetry of the gauge field  $b$ . The fermionic side also has a (different) manifest  $\frac{SU(2) \times U(1)}{Z_2}$  symmetry.

How do these symmetries act on physical operators? The operators  $(z^\dagger \sigma_x z, z^\dagger \sigma_y z, z^\dagger \sigma_z z)$  form a spin-1 representation of the  $SU(2)$  flavor symmetry in the  $NCCP^1$  model, while the operators  $(z_1^* z_2, \mathcal{M}_b)$  form a spin-1/2 representation under the flavor  $SU(2)$  symmetry of the  $QED_3$  theory ( $\mathcal{M}_b$  is the monopole operator in the  $CP^1$  model). It is easy to see that these together form a vector representation

$$(2\text{Re}\mathcal{M}_b, 2\text{Im}(\mathcal{M}_b), z^\dagger \sigma_x z, z^\dagger \sigma_y z, z^\dagger \sigma_z z) \sim (n_1, n_2, n_3, n_4, n_5) \quad (65)$$

of an enlarged  $SO(5)$  symmetry.

<sup>17</sup> Strictly speaking, when  $\lambda > 0$ , we need to compensate the anisotropy term with a mass term  $-\alpha\lambda(|z_1|^2 + |z_2|^2)$  (likewise for the  $w$ -theory), to keep the theories on both sides critical. We discuss this further in Sec. IV E.

A different and interesting perspective is provided by taking the proposed strong self-duality of the  $SU(2)$  invariant  $NCCP^1$  model as our logical starting point. In the original model, the operators  $(z^\dagger \sigma_x z, z^\dagger \sigma_y z, z^\dagger \sigma_z z)$  form a vector under a global  $SO(3)$  symmetry. In the dual model, the operators  $(w^\dagger \sigma_x w, w^\dagger \sigma_y w, z^\dagger \sigma_z w)$  form a vector under another global  $SO(3)$ . Since  $z^\dagger \sigma_z z \sim w^\dagger \sigma_z w$ , it is easy to see that the operators  $(w^\dagger \sigma_x w, w^\dagger \sigma_y w, z^\dagger \sigma_x w, z^\dagger \sigma_y w, z^\dagger \sigma_z z)$  form a vector under an enlarged  $SO(5)$  symmetry. Thus, the strong self-duality of the  $SU(2)$  invariant  $NCCP^1$  model implies the presence of an enlarged  $SO(5)$  symmetry. Conversely, the numerical evidence for the emergence of  $SO(5)$  symmetry at the Néel-VBS transition may be taken as support for the proposed self-duality of the  $SU(2)$ -invariant  $NCCP^1$  model.

Further support comes from considering a deformation of the model to reach the easy plane model. In the  $z$ -theory, this can be accomplished by perturbing with the operator  $(z^\dagger \sigma_z z)^2$ . In the  $w$ -theory the same operator is represented as  $(w^\dagger \sigma_z w)^2$ . We expect that this is a relevant perturbation (see Sec. IV D). The resulting flow leads directly to the self-duality of the easy plane  $NCCP^1$  model which has been independently derived (in its weak form). This is a good consistency check on the self duality of the  $SU(2)$  invariant model.

The action of discrete symmetries is similar to the easy plane case discussed above. Note, however, that the  $\mathcal{S}$  symmetry is a subgroup of the  $SU(2)$  flavor symmetry of the  $NCCP^1$  theory, and that  $\mathcal{S}_\psi$  is a subgroup of the  $SU(2)$  flavor symmetry of QED<sub>3</sub>. We, thus, do not need to consider them separately. For square lattice spin-1/2 magnets, the action of time reversal and lattice symmetries may be readily inferred from the easy plane case once we recognize that the extra field  $\phi$  transforms identically to  $z^\dagger \sigma_z z$ .

#### D. Allowed symmetry breaking terms

We now study the  $SO(5)$ -breaking operators that are allowed by microscopic symmetries. These operators must be irrelevant in order for the dualities and the emergent symmetries to hold in the IR without fine-tuning. Again we point out that in the situation where there is no fixed point with the higher symmetry, there could still be a pseudocritical regime up to a large but finite lengthscale  $\xi$  (see Sec. IX); in this case the requirements should be interpreted in terms of the effective scaling dimensions in this regime.

The putative emergent symmetry for the  $NCCP^1$  model and QED-Gross-Neveu is  $SO(5)$ . A natural guess is that the leading scalar operators, apart from the  $SO(5)$  vector  $n_a$  defined in Eq. (65),

are those in the two and four-index symmetric tensor representations of  $SO(5)$ . We denote these  $X_{ab}^{(2)}$  and  $X_{abcd}^{(4)}$ . At the level of symmetry,

$$X_{ab}^{(2)} \sim n_a n_b - \delta_{ab} n^2 / 5, \quad X_{abcd}^{(4)} \sim n_a n_b n_c n_d - (\dots), \quad (66)$$

where the subtraction  $(\dots)$  makes the operator traceless.  $X^{(2)}$  is certainly relevant.<sup>18</sup> The microscopic symmetries of  $NCCP^1$  allow the perturbation  $\sum_{a=3}^5 X_{aa}^{(2)}$ , which is an anisotropy between Néel and VBS ( $\sim \frac{2}{5}[n_3^2 + n_4^2 + n_5^2] - \frac{3}{5}[n_1^2 + n_2^2]$ ) and QED-GN allows  $X_{55}^{(2)}$ , corresponding to the mass term for the scalar field  $\phi$ . Since these are the perturbations that are tuned away to reach the critical point, they do not pose a problem for stability. However stability does require the irrelevance of  $X^{(4)}$ , as this gives rise to further symmetry-allowed perturbations. The symmetries of  $NCCP^1$  allow the higher Néel-VBS anisotropy  $\sum_{a=1}^2 \sum_{b=3}^5 X_{aabb}^{(4)}$ . For a quantum antiferromagnet on the square lattice,<sup>19</sup> the anisotropy  $\sum_{a=1}^2 X_{aaaa}^{(4)}$ , which breaks the  $U(1)$  symmetry for the VBS down to  $Z_4$ , is also allowed. In QED-GN the anisotropy  $X_{5555}^{(4)}$  is allowed.

Stability also requires the irrelevance of all  $SO(5)$ -singlet scalar operators. As discussed in Sec. IX, this requirement is in tension with conformal bootstrap results [24, 25]. However, the numerical evidence for  $SO(5)$  suggests that there is at least a pseudocritical regime where allowed  $SO(5)$ -breaking perturbations, including  $X^{(4)}$ , are effectively irrelevant.

## E. Phase diagram

We now discuss the phase diagram of the quantum magnet near the  $SU(2)$ -invariant deconfined critical point, allowing for a perturbation that breaks the spin symmetry to easy-plane. We assume the emergence of  $SO(5)$  symmetry at the  $SU(2)$  critical point.

It is useful to organize perturbations into representations of the  $SO(5)$  symmetry. The two perturbations that we must consider live in the symmetric tensor representation  $X^{(2)}$  of  $SO(5)$  discussed above (we drop the superscript),

$$X_{ab} \sim n_a n_b - \frac{\delta_{ab}}{5} n^2, \quad (67)$$

<sup>18</sup> See [15] for numerical results for correlation functions of  $X^{(2)}$  components.

<sup>19</sup> For the quantum antiferromagnet we are also allowed the term  $(\partial_x n_1)^2 + (\partial_y n_2)^2$  which locks VBS rotations to spatial rotations. The emergence of  $U(1)$  symmetry for the VBS in JQ model simulations [6, 8] implies that this term is also irrelevant. (On its own leaves the combined internal and spatial rotation unbroken, but this residual symmetry is broken by the torus boundary conditions used in simulations.) This term is absent for the model of [14, 15] since this model is isotropic in space-time.

and we denote them

$$\mathcal{O}_1 = X_{11} + X_{22}, \quad \mathcal{O}_2 = X_{55}. \quad (68)$$

First, the leading perturbation allowed in an  $SU(2)$ -symmetric spin model is  $\delta\mathcal{L} = \lambda_1\mathcal{O}_1$ , which drives the system into the Néel ordered phase for  $\lambda_1 < 0$  and into the VBS phase for  $\lambda_1 > 0$ . Second, breaking the spin symmetry down to that of the easy plane model allows the anisotropy  $\delta\mathcal{L} = \lambda_2\mathcal{O}_2$ . Again we already know the effect of this operator on its own: it drives the system into an easy-axis-ordered gapped phase for  $\lambda_2 > 0$ , and to the Néel-VBS phase transition of the easy plane model, potentially with  $O(4)$  symmetry, for  $\lambda_2 < 0$ .

The full phase diagram for small  $(\lambda_1, \lambda_2)$  follows by  $SO(5)$  symmetry, assuming that the only ordered phases in the vicinity of the  $SU(2)$  critical point are those mentioned above. Essentially, each of the three ordered phases is determined by which components of  $n_a$  are favored by the potential  $\lambda_1(X_{11} + X_{22}) + \lambda_2 X_{55}$ . More formally, the transition line  $\lambda_1 = \lambda_2 > 0$  between the Ising and VBS ordered phases may be fixed by noting that it corresponds to the perturbation

$$\lambda_1(X_{11} + X_{22} + X_{55}) = -\lambda_1(X_{33} + X_{44}), \quad (69)$$

where we have used the tracelessness of  $X$ . This is related to the Néel ordered line  $\lambda_1 < 0, \lambda_2 = 0$  by the  $SO(5)$  rotation  $n_1 \leftrightarrow n_3, n_2 \leftrightarrow n_4$ , which is precisely the self-duality of the  $NCCP^1$  theory. The phase diagram in the  $(\lambda_1, \lambda_2)$  basis is shown in Fig. 2.

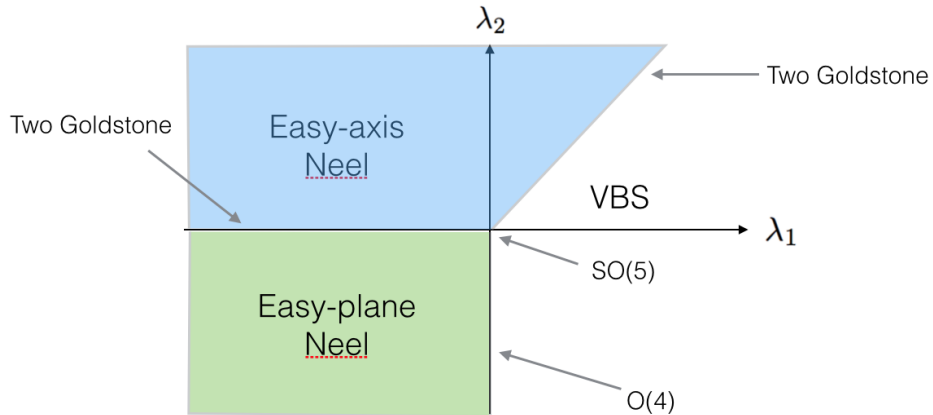


FIG. 2: Phase diagram near the  $SO(5)$ -invariant fixed point with perturbation of the form  $\lambda_1(X_{11} + X_{22}) + \lambda_2 X_{55}$ .

If we neglect perturbations that are irrelevant at the  $SO(5)$  fixed point, the transition between VBS and easy-axis order is governed by a Lagrangian with an emergent  $O(3)$  symmetry rotating  $(n_1, n_2, n_5)$ . This is spontaneously broken to  $SO(2)$  once  $\lambda_1 = \lambda_2$  flows to an order one value, yielding a pair of Goldstone modes. In reality, these Goldstone modes are only approximate if the bare  $\lambda_1 = \lambda_2$  is finite; the emergent  $O(3)$  is explicitly broken by dangerously irrelevant higher anisotropies<sup>20</sup> which are allowed by the symmetries of the lattice model. However these anisotropies will appear only at a parametrically large lengthscale when the bare  $\lambda_1 = \lambda_2$  is small.

Similarly, the line  $\lambda_1 = 0, \lambda_2 < 0$  which leads to the easy-plane deconfined transition has an emergent  $O(4)$  symmetry when higher anisotropies are neglected. Here however it is possible that the  $O(4)$  symmetry survives to asymptotically long lengthscales: this depends on the ultimate fate of the easy-plane theory.

The structure of the phase diagram above could be tested numerically. The most basic test is that the phase boundaries all meet at nonzero angles, showing that the distinct components of  $X$  have the same scaling dimension.<sup>21</sup> There is also universal information in the slopes of the phase boundaries. In the microscopic model a more natural basis for perturbations is  $\delta\mathcal{L} = \tilde{\lambda}_1\mathcal{O}_1 + \tilde{\lambda}_2\tilde{\mathcal{O}}_2$ , where  $\mathcal{O}_1$  is the lattice operator which drives the Néel-VBS transition and  $\tilde{\mathcal{O}}_2 \sim X_{55} + \frac{1}{3}(X_{11} + X_{22})$  is a modified easy-plane anisotropy. The numerical coefficient in the latter is fixed by demanding that it belongs to the traceless symmetric tensor representation of spin  $SO(3)$ :  $\mathcal{A}_{ij} \sim X_{ij} + \frac{1}{3}(X_{11} + X_{22})$ , where  $i, j = 3, 4, 5$ . In  $NCCP^1$ ,  $\mathcal{O}_1 \sim -|z|^2$  and  $\tilde{\mathcal{O}}_2 \sim [|z_1|^2 - |z_2|^2]^2 - \frac{1}{3}[|z_1|^2 + |z_2|^2]^2$ . When we draw the phase diagram in the  $(\tilde{\lambda}_1, \tilde{\lambda}_2)$  plane, the easy-axis Néel/VBS transition line is at  $\tilde{\lambda}_1 = 2c\tilde{\lambda}_2$ , and the easy-plane Néel/VBS transition line is at  $\tilde{\lambda}_1 = -c\tilde{\lambda}_2$ . The constant  $c > 0$  is arbitrary since the normalization of the lattice operators is arbitrary, but the ratio of the slopes of the two lines is a fixed constant which could be checked numerically. The phase diagram in this  $(\tilde{\lambda}_1, \tilde{\lambda}_2)$  basis is shown schematically in Fig. 3.

Alternately we may check universal amplitudes using correlation functions, once the location of one of the nontrivial transition lines is determined. Let us normalize  $X_{ab}$  so that

$$\langle X_{ab}(x)X_{cd}(0) \rangle = \frac{1}{2} \left( \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \frac{2}{5}\delta_{ab}\delta_{cd} \right) \frac{1}{x^{2\Delta_2}}, \quad (70)$$

where  $\Delta_2$  is the scaling dimension of  $X_{ab}$ . Assume that we can identify (numerically) either the

<sup>20</sup> Components of  $X^{(4)}$  in the notation of Sec. IV D. These are presumed to be irrelevant at the  $SU(2)$  critical point, but are relevant in the  $O(3)$ -breaking phase.

<sup>21</sup> See Ref. [15] for a direct numerical test of this using correlation functions.

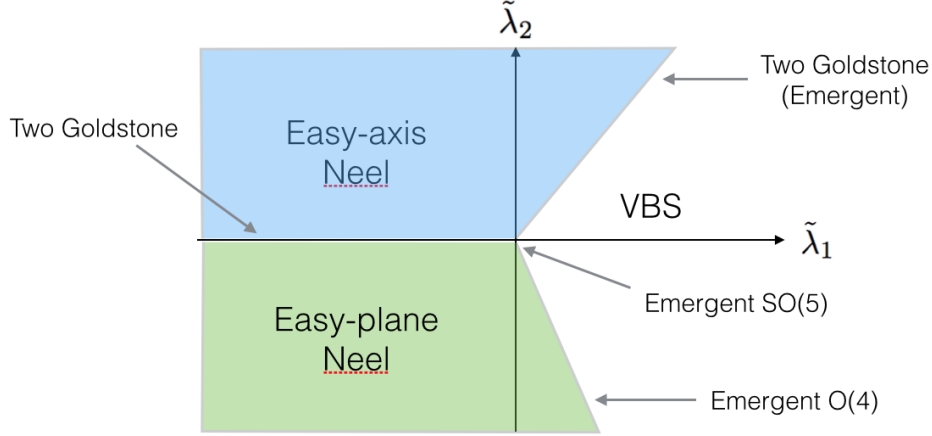


FIG. 3: Phase diagram near the  $SO(5)$ -invariant fixed point with perturbation of the form  $-\tilde{\lambda}_1(|z_1|^2 + |z_2|^2) + \tilde{\lambda}_2[(|z_1|^2 - |z_2|^2)^2 - \frac{1}{3}(|z_1|^2 + |z_2|^2)^2]$ . This is the natural perturbation to consider in the context of deconfined criticality in quantum magnets. The emergent  $SO(5)$  symmetry requires that the slope of the lower transition line is twice that of the upper transition line.

perturbation  $\mathcal{O}_2 \sim -X_{55}$  which drives the system along the VBS/easy-plane Néel phase boundary, or the perturbation  $\mathcal{O}_3 \sim X_{11} + X_{22} + X_{55}$  that drives the system to the first-order transition between the VBS and the easy-axis Néel state. Then by Eq. (70) the following statements, independent of normalization, should be true:

$$\frac{\langle \mathcal{O}_1(x) \mathcal{O}_2(0) \rangle^2}{\langle \mathcal{O}_1(x) \mathcal{O}_1(0) \rangle \langle \mathcal{O}_2(x) \mathcal{O}_2(0) \rangle} = \frac{1}{6}, \quad \frac{\langle \mathcal{O}_1(x) \mathcal{O}_3(0) \rangle^2}{\langle \mathcal{O}_1(x) \mathcal{O}_1(0) \rangle \langle \mathcal{O}_3(x) \mathcal{O}_3(0) \rangle} = \frac{4}{9}. \quad (71)$$

Similar tests are possible in the QED-Gross-Neveu theory, if the fixed point is found. There  $\mathcal{O}_2 \sim \phi^2$  is the Ising mass operator that drives the system through the Gross-Neveu transition between  $QED_3$  and the gapped phase in which  $\phi$  has condensed. The fermion chiral mass  $\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2$  is a mixture of  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . The  $SU(2)$  flavour symmetry of QED requires the chiral mass to be orthogonal to  $\mathcal{O}_2$ , so by Eq. (70)

$$\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2 \sim \mathcal{O}_1 + \frac{1}{2}\mathcal{O}_2. \quad (72)$$

Now if we consider a perturbation of the form  $m_\phi\phi^2 + m_\psi(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2)$ , the phase diagram will look like Fig. 4. The phase diagram is symmetric under the reflection across the  $m_\phi$  axis simply because of the fermion flavor symmetry. The two transitions lines near the gapped phase are given

by  $m_\phi = \pm \frac{1}{2c'} m_\psi > 0$  with  $c' > 0$  being a normalization-dependent constant. So  $c'$  alone does not provide nontrivial information. However, it enters into the ratio of correlation functions:

$$\frac{\langle \phi^2(x) \phi^2(0) \rangle}{\langle (\bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2)(x) (\bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2)(0) \rangle} = \frac{4}{5} (c')^2, \quad (73)$$

which is in principle testable.

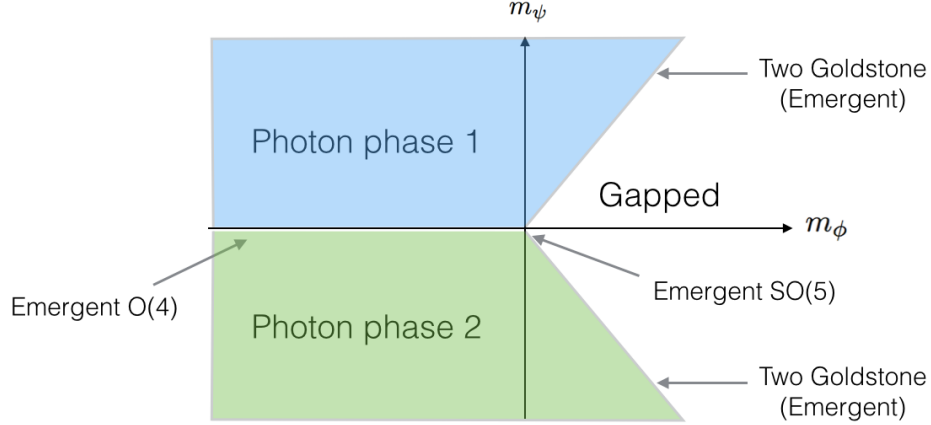


FIG. 4: Phase diagram near the  $SO(5)$ -invariant fixed point with perturbation of the form  $m_\phi \phi^2 + m_\psi (\bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2)$ . This is the natural perturbation to consider in the context of QED<sub>3</sub>–Gross-Neveu theory. The emergent  $SO(5)$  symmetry predicts that the slope of the transition lines is related to the relative amplitude of the correlation functions of the two operators, Eq. (73)

We end this section with a discussion on the nature of the transitions at  $m_\phi = \pm \frac{1}{2c'} m_\psi > 0$  in Fig. 4. As discussed already, we expect the two transitions to be direct first order, instead of broadening into coexistence phases. How do we understand this from a fermion mean-field point of view? We can calculate the mean field free energy with respect to  $m_\psi$  and  $\langle \phi \rangle$ , treating the fermions as almost non-interacting, which is valid when the fermion flavor number  $N_f \rightarrow \infty$ . The result is proportional to

$$E \sim |m_\psi + \langle \phi \rangle|^3 + |m_\psi - \langle \phi \rangle|^3. \quad (74)$$

Interestingly, this function gives no preference to either scenario (first order or coexistence). Presumably fluctuations beyond mean-field will break this degeneracy and lead to a direct transition.



## V. BULK INTERPRETATION: $U(1) \times U(1)$ THEORY

From the sigma model point of view, it is known that the symmetries of the field theories discussed so far have to be anomalous [53, 67]. Deconfined criticality can nevertheless be realized in quantum magnets because lattice rotation symmetries are not on-site, and therefore can be implemented in a seemingly anomalous fashion in the continuum theory. If we want all the symmetries to be on-site, the theories must be regularized on the boundary of a 3 + 1D bulk. In this section we discuss the bulk physics corresponding to the easy-plane deconfined critical point. This provides considerable insight into the duality web and the unconventional symmetry actions of the theory.

In Eq. (36) we see that the spin-flip  $\mathcal{S}$ , time reversal, and other symmetries are anomalous:

$$\mathcal{L} \rightarrow \mathcal{L} - \frac{1}{2\pi} B_1 dB_2. \quad (75)$$

We will initially focus on the  $\mathcal{S}$  symmetry. This anomaly is natural from the sigma model approach: the  $\mathcal{S}$  symmetry is an improper  $O(4)$  rotation  $(n_1, n_2, n_3, n_4) \rightarrow (n_1, -n_2, n_3, n_4)$  which fixes  $\theta = \pi$ , and such a symmetry is typically expected to be anomalous.

The anomaly can be cured by placing the 2 + 1D theory at the boundary of a 3 + 1D bosonic SPT insulator with  $[U(1) \rtimes Z_2] \times U(1)$  symmetry. Let us couple 3 + 1D background gauge fields  $B_1$  and  $B_2$  to the two  $U(1)$  symmetries such that under  $\mathcal{S}$  they transform as

$$\mathcal{S} : B_1 \rightarrow -B_1, B_2 \rightarrow B_2. \quad (76)$$

A non-trivial SPT phase of such a bosonic system then has a response characterized by a mutual  $\Theta$  term at  $\Theta = \pi$  for the two gauge fields  $B_1$  and  $B_2$  of the form

$$-\frac{\Theta}{(2\pi)^2} \int_{Bulk} dB_1 \wedge dB_2, \quad \Theta = \pi \quad (77)$$

Notice that under  $\mathcal{S}$ ,  $\Theta \rightarrow -\Theta$ , and therefore  $\Theta = \pi$  is protected<sup>22</sup> by the  $\mathcal{S}$  symmetry.

Now consider the surface of this boson SPT phase. The bulk  $\Theta$  term leads to a surface state with anomalous symmetry realization. Clearly this anomaly is exactly the same as in the 2 + 1D easy plane  $NCCP^1$  field theory, Eq. (36). Specifically if we add a bulk contribution,  $-\frac{1}{4\pi} B_1 dB_2$ , to the Lagrangian, defining

$$\mathcal{L}_{z,SPT} = \mathcal{L}_z - \frac{1}{4\pi} B_1 dB_2, \quad (78)$$

<sup>22</sup> We have used the periodicity of the physics under shifts  $\Theta \rightarrow \Theta + 2\pi$  which holds for this system.

(where the extra term, which is not well-defined as a mutual Chern-Simons term in pure  $2+1$ D, is really a short-hand for the bulk mutual  $\Theta$ -term at  $\Theta = \pi$ ) it is easy to check that  $\mathcal{L}_{z,SPT}$  is indeed invariant under the spin-flip symmetry.

Now imagine gauging the  $U(1)_{B_1} \times U(1)_{B_2}$  symmetry in the bulk. An important observation is that due to the mutual  $\Theta$ -term the monopole of one species carries charge  $\pm 1/2$  of the *other* species. Let us label the charge-monopole lattice by  $(q_{e1}, q_{e2}; q_{m1}, q_{m2})$ . Here  $(q_{e1}, q_{m1})$  are the electric and magnetic charges under  $U(1)_{B_1}$ , and so on. The mutual  $\Theta$  term implies the relations

$$q_{e2} = \frac{q_{m1}}{2} \pmod{Z}, \quad q_{e1} = \frac{q_{m2}}{2} \pmod{Z}. \quad (79)$$

Note that  $q_{m1}, q_{m2} \in Z$ .

There is a correspondence between fields in the boundary theory and particles in the bulk theory: bulk electric charges correspond to electrically charged surface fields, and bulk magnetic charges correspond to vortices on the surface. For the ‘physical’ bosons the correspondence is  $\Phi_{B_1} \sim (1, 0; 0, 0)$  and  $\Phi_{B_2} \sim (0, 1; 0, 0)$ . The surface fields  $z_{1,2}$  are vortices in  $\Phi_{B_2}$ , and they carry charge  $q_{B_1} = \pm 1/2$ . Their bulk avatars are thus the dyons  $z_{1,2} \sim (\pm \frac{1}{2}, 0; 0, 1)$ .

The bosonic self-duality of the easy plane  $NCCP^1$  theory leads to a description in terms of complex fields  $w_{1,2}$  which are vortices of  $\Phi_{B_1}$  and carry charges  $q_{B_2} = \pm 1/2$ . Clearly their bulk avatars are  $w_{1,2} \sim (0, \pm \frac{1}{2}; 1, 0)$ . The surface self-duality is thus connected to the obvious bulk duality between descriptions in terms of these two sets of dyons.

Consider the bound states of these two kinds of dyons with quantum numbers

$$\left(\frac{1}{2}, \frac{1}{2}; 1, -1\right), \quad \left(-\frac{1}{2}, -\frac{1}{2}; 1, -1\right). \quad (80)$$

These are both fermions. We identify them as the bulk avatars of  $\psi_{1,2}$ . This can be confirmed directly from the surface theory. Consider the  $QED_3$  theory with  $\psi_{1,2}$  fermions in Eq. (22) with the added ‘bulk’ contribution  $-\frac{1}{4\pi} B_1 dB_2$ . Notice that  $B = (B_1 + B_2)/2$ . We see that the  $\psi_{1,2}$  fermions indeed have the right charges and vorticities to correspond to these bulk fermionic dyons.

Thus the duality between the easy plane  $NCCP_1$  theory and the  $QED_3$  theory can be understood in terms of a bulk duality that trades the bosonic  $(\pm \frac{1}{2}, 0; 0, 1)$  particles with the fermionic particles of Eq. (80).

What about the dual fermions  $\chi_{1,2}$ ? They correspond to

$$\left(\frac{1}{2}, -\frac{1}{2}; 1, 1\right), \quad \left(-\frac{1}{2}, \frac{1}{2}; 1, 1\right). \quad (81)$$

Indeed this is exactly what is implied by the dual fermionic surface theory.

The fermion-fermion duality of  $QED_3$  can thus be related to a corresponding bulk fermion-fermion duality of the  $U(1) \times U(1)$  gauge theory.

Notice that  $(q_{e1}, q_{e2}; q_{m1}, q_{m2}) \rightarrow (-q_{e1}, q_{e2}; -q_{m1}, q_{m2})$  under  $\mathcal{S}$ . It is immediately clear that the two fermionic dyons corresponding to  $\psi_{1,2}$  become the two dyons corresponding to  $\chi_{1,2}$  under  $\mathcal{S}$ . This offers a bulk interpretation of the non-trivial action of  $\mathcal{S}$  on the surface  $QED_3$  theory, which exchanges  $\psi_{1,2}$  and their dual fermions  $\chi_{1,2}$ .

Likewise, under the fermion flavor exchange symmetry  $\mathcal{S}_\psi$ , which acts as  $B_1 \leftrightarrow -B_2$ , the dyons corresponding to  $z_{1,2}$  and  $w_{1,2}$  are exchanged. This is a simple bulk picture of the symmetry action in the boundary  $NCCP^1$  model, which is implemented through the self-duality transform.

Note that the  $U(1) \times U(1)$  gauge theory has an  $Sp(4, Z)$  invariance corresponding to basis changes in the four dimensional charge-monopole lattice. This is because the basis change must preserve the area of the unit cell of each two dimensional subspace corresponding to each of the two  $U(1)$  gauge theories. The surface web of dualities we have discussed can be understood as the effects of various  $Sp(4, Z)$  transformations of the bulk gauge theory.

We should also emphasize that the bulk duality offers a simple picture of the surface duality, but does not prove the surface duality between IR fixed point theories.

Now let us turn briefly to time reversal which acts on  $B_{1,2}$  as

$$T(B_{1,2}) = B_{1,2}. \quad (82)$$

Under this the bulk  $\Theta$  term is odd, but as before  $\Theta = \pi$  is time reversal symmetric. Correspondingly when a surface is present, the contribution from this bulk  $\Theta$  term will exactly cancel the time-reversal anomaly of the surface theories. The bulk charges transform under time reversal as

$$T : (q_{e1}, q_{e2}; q_{m1}, q_{m2}) \rightarrow (-q_{e1}, -q_{e2}; q_{m1}, q_{m2}). \quad (83)$$

It is readily checked that this is precisely consistent with the time reversal action on each of the surface theories.

We already saw that the translation  $\tilde{T}_y$  can be related to a combination of  $\mathcal{S}$  and  $\mathcal{S}_\psi$  and therefore does not need separate discussion.

## VI. THEORIES WITH MANIFEST $SO(5)$ SYMMETRY

So far none of our field-theoretic descriptions of the deconfined critical point possessed explicit  $SO(5)$  symmetry in the UV: this symmetry at best emerged in the IR. An exception was the  $SO(5)$  NLSM with a WZW term at level 1; however, this model is non-renormalizable, so while one can infer symmetry information from it, strictly speaking its dynamics in the disordered phase is not well-defined. Here we present two renormalizable theories with explicit  $SO(5)$  symmetry, namely  $N_f = 2$   $QCD_3$  and its Higgs descendent  $N_f = 4$  compact  $QED_3$ . While the IR fates of these theories are unknown, they share the same anomaly with the deconfined critical point. So there is the possibility (among others) that either of them may flow to the deconfined critical point.

### A. Parton construction of $N_f = 2$ $QCD_3$

To see the connection between deconfined criticality and these theories, we now review the construction of the  $\pi$ -flux state on the square lattice [68] and demonstrate that its low energy theory,  $QCD_3$ , has an emergent  $SO(5)$  symmetry. The Neel and VBS order parameters transform, as expected, as the 5 components of an  $SO(5)$  vector.

Consider the standard fermionic parton decomposition:

$$\mathbf{S}_i = \frac{1}{2} f_{i\alpha}^\dagger \sigma_{\alpha\beta} f_{i\beta}, \quad f_{\alpha i}^\dagger f_{\alpha i} = 1, \quad (84)$$

where  $i$  labels a lattice site and the spin indices  $\alpha, \beta$  are summed over. Let us form a matrix

$$X_i = \begin{pmatrix} f_{i\uparrow} & -f_{i\downarrow}^\dagger \\ f_{i\downarrow} & f_{i\uparrow}^\dagger \end{pmatrix}. \quad (85)$$

The decomposition (84) is invariant under local  $SU(2)$  gauge rotations

$$SU(2)_g : \quad X_i \rightarrow X_i (U_i^g)^\dagger. \quad (86)$$

The physical  $SU(2)$  spin rotations act as

$$SU(2)_s : \quad X_i \rightarrow U^s X_i \quad (87)$$

and we can rewrite (84) as  $\mathbf{S}_i = \frac{1}{4} \text{tr}(X_i^\dagger \sigma X_i)$ . It will occasionally be convenient to write  $f_\uparrow = \frac{1}{\sqrt{2}}(\chi_0 + i\chi_3)$ ,  $f_\downarrow = \frac{1}{\sqrt{2}}(-\chi_2 + i\chi_1)$ , i.e  $X = \frac{1}{\sqrt{2}}(\chi_0 + i\chi_a \sigma^a)$ , where  $\chi_m$ ,  $m = 0, 1, 2, 3$ , are Majorana fermions.  $\chi_m$  transforms as an  $SO(4)$  vector under the combined action of  $SU(2)_s$  and  $SU(2)_g$ .

We consider a mean-field state on the square lattice,

$$H_{MF} = -\frac{i}{2} \sum_{ij} t_{ij} \chi_{im} \chi_{jm} \quad (88)$$

where  $t_{ij} = -t_{ji}$ , and  $t_{i+\hat{x},i} = 1$ ,  $t_{i+\hat{y},i} = (-1)^{i_x}$  so that there is  $\pi$  flux through each plaquette. This mean-field explicitly preserves the  $SU(2)_s$  and  $SU(2)_g$  symmetries, while lattice symmetries now act in a projective manner (see below). Each flavor  $m$  of Majorana fermions produces two gapless Majorana cones, so the low-energy theory becomes,

$$H_{MF} = i \sum_{v=1,2} \chi_{m,v} (\tau^x \partial_x - \tau^z \partial_y) \chi_{m,v} \quad (89)$$

with  $\tau$  acting on suppressed sublattice indices  $\sigma \in A, B$  (see below). The index  $v$  runs over two valleys and lattice fields are related to continuum ones in the following way. The unit cell is doubled by  $t_{ij}$ . We label sites with even  $i_x$  by  $A$  and odd  $i_x$  by  $B$ , and label unit cells by  $\bar{i} \in (2\mathbb{Z} + 1/2, \mathbb{Z})$ . Letting  $\chi_{\bar{i}} = (\chi_{\bar{i}-\hat{x}/2,A}, \chi_{\bar{i}+\hat{x}/2,B})$ ,

$$\chi_{m,\bar{i}} \sim \tau^x \chi_{m,v=1}(x) + (-1)^{\bar{i}_y} \chi_{m,v=2}(x). \quad (90)$$

We can rewrite the mean-field Lagrangian as

$$\mathcal{L}_{MF} = i \bar{\chi}_{v,m} \gamma^\mu \partial_\mu \chi_{v,m}, \quad (91)$$

where  $\bar{\chi} = \chi^T \gamma^0$ ,  $\gamma^0 = \tau^y$ ,  $\gamma^x = i\tau^z$ ,  $\gamma^y = i\tau^x$ . The action of lattice symmetries is:

$$T_x : \chi \rightarrow \mu^x \chi, \quad (92)$$

$$T_y : \chi \rightarrow \mu^z \chi, \quad (93)$$

$$R_{\pi/2,A} : \chi \rightarrow e^{\pi i \tau^y/4} e^{-\pi i \mu^y/4} \chi(-y, x), \quad (94)$$

$$P_{x,A} : \chi \rightarrow \tau^z \mu^z \chi(-x, y), \quad (95)$$

$$T : \chi \rightarrow \tau^y \mu^y \chi, \quad i \rightarrow -i, \quad (96)$$

where  $\mu$  acts on the valley index  $v$  and spin/color indices  $m$  have been suppressed. The subscript  $A$  on  $\pi/2$  rotation  $R$  and reflection  $P$  indicates that these are around an  $A$  site. These symmetries prohibit any quadratic term in  $\chi$  with no derivatives in  $\mathcal{L}_{MF}$ .

The mean-field theory has an  $O(8)$  symmetry acting on  $m, v$ . However this is broken by fluctuations of  $SU(2)_g$  gauge field and four-fermi interactions. Let us first focus on the gauge field fluctuations. For this purpose it is convenient to introduce a  $4 \times 2$  matrix,  $X_{\alpha,v;\beta}$  via

$$X_{\alpha,v;\beta} = \frac{1}{\sqrt{2}} (\chi_{0,v} \delta_{\alpha\beta} + i \chi_{a,v} \sigma_{\alpha\beta}^a). \quad (97)$$

The sublattice index is suppressed above. The hermiticity of  $\chi$  imposes an important relation

$$X^* = \sigma^y X \sigma^y. \quad (98)$$

SU(2) spin and SU(2) gauge transformations act on  $X$  from the left and right. The covariant derivative with respect to the dynamical SU(2)<sub>g</sub> gauge field  $a$  acts on  $X$  as  $D_\mu^a X = \partial_\mu X + iX a_\mu$ , and  $a$  transforms as

$$\text{SU}(2)_g : X \rightarrow XU_g^\dagger, \quad a_\mu \rightarrow U_g a_\mu U_g^\dagger - i\partial_\mu U_g U_g^\dagger. \quad (99)$$

The Lagrangian including the dynamical gauge field then is

$$\mathcal{L}_{QCD_3} = i\text{tr}(\bar{X}\gamma^\mu D_\mu^a X), \quad (100)$$

with  $\bar{X} = X^\dagger \gamma^0$ . We see that (100) is invariant under a global symmetry,

$$\text{Sp}(4) : X \rightarrow LX \quad (101)$$

with  $L \in \text{Sp}(4)$  - a unitary matrix acting on spin/valley indices  $\alpha, v$  of  $X_{\alpha,v;\beta}$ . The fact that  $L$  lies in  $\text{Sp}(4)$ , i.e.  $L^T \sigma^y L = \sigma^y$ , instead of in the larger group  $U(4)$  comes from the reality condition (98). The lattice symmetries in (96) are elements of this  $\text{Sp}(4)$  global symmetry combined with spatial symmetries of Dirac theory. We note that the global symmetry  $\text{Sp}(4)$  and the gauge group  $\text{SU}(2)_g$  share a common non-trivial element: the center  $-1$ . Thus, the physical global symmetry after modding out by  $\text{SU}(2)_g$  is actually  $\text{Sp}(4)/Z_2 = \text{SO}(5)$  (it is useful to recall that  $\text{Sp}(4) = \text{Spin}(5)$ ). An order parameter for this  $\text{SO}(5)$  symmetry is given by a five component vector:

$$n^a = \text{tr}(\bar{X}\Gamma^a X) \quad (102)$$

with  $\Gamma = \{\mu^z, -\mu^x, \sigma^x \mu^y, \sigma^y \mu^y, \sigma^z \mu^y\}$ . The first two components,  $n^1, n^2$  have precisely the transformation properties of the  $x$  and  $y$  components of the VBS order parameter, while the last three components  $\text{tr}(\bar{X}\sigma^a \mu^y X)$  correspond to the Néel order parameter.

We note in passing that if we want to be less explicit about the full emergent symmetry of (100), we can express the Lagrangian in terms of two flavors of SU(2)<sub>g</sub> charged complex Dirac fermions,  $\psi_{\alpha,v} = i\sigma_{\alpha,\beta}^y X_{1,v;\beta}$ , with  $\alpha$  being the color index, and

$$\mathcal{L} = i\bar{\psi}_v \gamma^\mu (\partial_\mu - ia_\mu) \psi_v \quad (103)$$

with  $\bar{\psi}_v = \psi_v^\dagger \gamma^0$ . In other words, this theory is  $N_f = 2$  QCD<sub>3</sub>.

There are (at least) three possible scenarios for this theory. First,  $N_f = 2$   $QCD_3$  could confine, and in the process spontaneously break  $SO(5)$  symmetry by generating a condensate  $\langle n^a \rangle \neq 0$ . In the setting of the spin system, quartic terms in the Lagrangian will then select either the VBS state or the Néel state. This is the boring scenario.

Second,  $N_f = 2$   $QCD_3$  could in principle flow to a stable gapless fixed point at which all perturbations (e.g. four-fermi couplings and velocity anisotropies) which preserve lattice and  $SO(3)_s$  symmetries are irrelevant. We would then have a completely stable gapless spin-liquid phase with emergent  $SO(5)$  symmetry. (In principle QCD could also flow to a gapped  $SO(5)$ -invariant spin liquid; as shown in Sec. VIE this is only possible if time reversal symmetry is broken.)

Third,  $N_f = 2$   $QCD_3$  could flow to a gapless fixed point which is stable in the presence of  $SO(5)$ , but which allows a single relevant perturbation when  $SO(5)$  is broken to the physical symmetry: the operator  $\mathcal{O}_1$  in Sec. IVE (breaking  $SO(5)$  to  $SO(3)_s \times SO(2)_{VBS}$ ). Then  $N_f = 2$   $QCD_3$  tuned to an  $SO(5)$  symmetric point describes the deconfined critical point, and perturbing it by  $\mathcal{O}_1$  drives it into either the VBS phase or the Néel phase. This is the scenario relevant for this paper.

### B. Higgs descendent: $N_f = 4$ compact $QED_3$

Starting from  $N_f = 2$   $QCD_3$ , we now Higgs the gauge group from  $SU(2)$  down to  $U(1)$ . We introduce and condense a scalar field  $\phi$  that transforms as a spin-1 vector under  $SU(2)_g$  and as a scalar under  $SO(5)$  (such a field is allowed in the theory). After a charge-conjugation redefinition of half of the Dirac fermions, the resulting theory is

$$\mathcal{L} = \sum_{i=1}^4 i\bar{\psi}_i \gamma^\mu (\partial_\mu - ia_\mu) \psi_i + (\lambda \mathcal{M}_a + h.c.), \quad (104)$$

where  $a_\mu$  is now a  $U(1)$  gauge field, and the monopole operator  $\mathcal{M}_a$  represents instanton events that change the flux of  $a_\mu$  by  $2\pi$ . In general such a term should be expected when the  $U(1)$  gauge field comes from Higgsing of a higher gauge symmetry. In condensed matter language<sup>23</sup> such theories are called compact  $QED_3$ .

The fermion fields  $\psi_i$  transform as a spinor representation under the global  $SO(5)$  – this follows simply from the symmetry properties of  $QCD_3$ . Naively one might expect the Lagrangian Eq. (104)

<sup>23</sup> The  $QED_3$  theory discussed here is not to be confused with the “staggered flux state” known in the spin liquid literature [68] (which is also described by  $QED_3$  with four Dirac fermions at low energy) — lattice symmetries act in distinct ways in these two theories.

to have a larger flavor symmetry, say  $SU(4)$ , respected by the Dirac term. However, it turns out that the monopole term breaks the symmetry down to  $SO(5)$ . This can be seen by analyzing the fermion zero-modes [55] associated with the monopole operator  $\mathcal{M}_{0a}$ : each Dirac fermion  $\psi_i$  contributes a complex fermion zero mode  $f_i$  in the monopole background, and gauge invariance requires two of the four zero modes to be filled in the ground state, so a gauge-invariant operator should be represented as  $f_i^\dagger f_j^\dagger \mathcal{M}_{0a}$ . There are in total six of them and it is straightforward to check that they split into  $6 = 1 \oplus 5$  with respect to the  $SO(5)$  symmetry. The monopole operator that appears in the Lagrangian in Eq. (104) is precisely the  $SO(5)$  scalar monopole. It transforms nontrivially under higher flavor symmetries, and  $SO(5)$  is the maximal flavor group that is compatible with it.

Since this  $N_f = 4$  compact  $QED_3$  is just a Higgs descendent of  $N_f = 2$   $QCD_3$ , they must have the same anomaly structure. Therefore they share the same set of possible IR behaviors, including those discussed at the end of Sec. VI A. Of course the two theories could pick different choices.

### C. Interpretation as surface theory of 3 + 1D boson SPT

Here we show that  $SO(5)$  symmetric  $N_f = 2$   $QCD_3$  (and hence  $N_f = 4$  compact  $QED_3$ ) can be interpreted as a surface theory of a bosonic  $SO(5)$ -protected 3 + 1D SPT phase. This statement is independent of the dynamics of  $QCD_3$ : it remains true even if the theory spontaneously breaks  $SO(5)$ . To make this statement, we have to understand how an  $SO(5)$  background gauge field  $\mathcal{A}^5$  enters in  $QCD_3$  and show that this theory has an anomaly, which is precisely compensated by the 3 + 1D SPT bulk. Here we establish this using a physical argument. In Sec. VII we provide a precise formal proof.

Let us first determine the anomaly of this theory by thinking about a background gauge field that couples to the  $SO(5)$  currents. An  $SO(5)$  gauge field  $\mathcal{A}^5$  admits  $Z_2$ -indexed monopole configurations, since  $\pi_1(SO(5)) = Z_2$ . In 2+1D these correspond to instanton events and we can ask whether there is anything non-trivial about them.

We examine a monopole background of the following form. Consider an  $SO(2) \times SO(3)$  subgroup of  $SO(5)$ , with  $SO(2)$  acting on the first two components of vector  $n^a$  in Eq. (102) and  $SO(3)$  on the last three. Place a unit magnetic monopole in the  $SO(2)$  subgroup, i.e.

$$\mathcal{A}_\mu^5 = \mathcal{A}_\mu^{mon}(x) T^1 \quad (105)$$

where  $T^1$  is the generator of  $SO(2)$  and  $\mathcal{A}^{mon}$  is the standard potential associated with a unit



magnetic monopole. In the presence of such a background  $\mathcal{A}^5$  field, only a subgroup of the global  $SO(5)$  symmetry survives: these are rotations in an  $SO(3)$  subgroup (whose generators commute with  $T^1$ ) and the  $SO(2)$  rotations generated by  $T^1$  itself.

The following argument provides a hint of the properties of the instanton. Rather than considering the  $QCD_3$  theory directly, suppose we add in an extra field  $\hat{n}$ , transforming in the vector representation of  $SO(5)$ , that couples to fermion bilinears through a Yukawa coupling. In the limit that this coupling is strong, we can integrate out the fermions and standard methods [26, 27, 69] produce an  $SO(5)$  non-linear sigma model in the  $\hat{n}$  field with a level-1 WZW term (Sec. II A). Now the  $SU(2)$  gauge field does not couple directly to any matter field, and is expected to confine at low energy, leaving behind the  $SO(5)$  WZW model as the remaining nontrivial theory. Indeed, this supports the idea that  $QCD_3$  correctly describes the Néel-VBS intertwinement in square lattice quantum magnets. Physically the  $SO(5)$  instanton in Eqn. (105) has the effect of creating a vortex in two components of the  $\hat{n}$  field. We now ask how this vortex transforms under the unbroken symmetry  $SO(2) \times SO(3)$ . The vortex carries no charge under  $SO(2)$ , but we know that the vortex transforms as a *spinor* under  $SO(3)$  due to the WZW term. We conclude that the instanton configuration described above transforms as an  $SO(3)$  spinor with zero  $SO(2)$  charge also.

For a conventional  $SO(5)$  sigma model (*i.e* without the WZW term) the  $SO(5)$  instanton will transform trivially under  $SO(2) \times SO(3)$ . The projective transformation of the instanton under the  $SO(3)$  subgroup tells us that in the presence of the level-1 WZW term the  $SO(5)$  symmetry is realized anomalously. It cannot be realized as the on-site symmetry of any strictly 2+1D model. Clearly the same instanton structure also characterizes the  $QCD_3$  theory.

This is the physics of the desired anomaly. Note also that this instanton operator is bosonic (*i.e* in relativistic parlance it has spin-0 under spatial rotations).

It is instructive to rederive the instanton structure of  $QCD_3$  directly from the UV Majorana fermion theory. We now briefly indicate how this works out. It is important to recognize that the fermions that enter  $QCD_3$  transform as a fundamental of  $Sp(4)$ , although the physical global symmetry is  $SO(5) = Sp(4)/Z_2$ . We therefore need to lift the  $SO(5)$  gauge field  $\mathcal{A}^5$  to an  $Sp(4)$  gauge field  $A^5$ , which enters the theory (100) via  $D_\mu X \rightarrow (\partial_\mu - iA_\mu^5)X + iXa_\mu$ . For instance, consider

$$A^5 = \frac{\mathcal{A}^{mon}}{2} \mu^y \quad (106)$$

where  $\mu$  are Pauli matrices that act on the valley index. Naively this may seem to require creating  $\pi$  flux through the non-zero component of  $A^5$  which apparently violates Dirac quantization for the

monopole. However, we should remember that we also have a dynamical  $SU(2)$  gauge field  $a$  that the fermions are coupled to: if the lift to  $Sp(4)$  is accompanied by a  $\pi$  flux instanton in one of the three components of the  $SU(2)$  gauge field, then we have a sensible configuration that satisfies Dirac quantization.<sup>24</sup> For instance, we may give the dynamical gauge field a background value  $a = a^3 \sigma^3$ , with

$$a^3 = \frac{\mathcal{A}^{mon}}{2}. \quad (107)$$

It is then convenient to rewrite Eq. (100) in terms of a single color component of  $X$ , e.g.  $X_{\alpha v, \uparrow}$ :

$$\mathcal{L} = i \bar{X}_{\alpha v, \uparrow} \gamma^\mu ((\partial_\mu + i a_\mu^3) \delta_{\alpha \alpha'} \delta_{v v'} - i A_{\alpha, v; \alpha', v'}^5) X_{\alpha' v', \uparrow}. \quad (108)$$

Observe that 2 Dirac fermions  $X_{\alpha-}$  with  $\mu^y = -1$  see a  $2\pi$  flux instanton, and another 2 Dirac fermions  $X_{\alpha+}$  with  $\mu^y = +1$  see no background flux (we drop the color index  $\uparrow$  here). Further, each pair of Dirac fermions transforms as a spin-1/2 under the global  $SO(3)_s$  subgroup left unbroken by the  $SO(5)$  monopole. In the language of the state-operator correspondence, in the presence of the instanton background the  $X_{\alpha-}$  fermions will form two zero modes. Charge neutrality with respect to the color gauge field  $a^3$  then implies that we occupy one of these zero modes. Thus, the instanton will transform as an  $SO(3)$  spinor in agreement with the arguments above. It is also easy to see that it has zero charge under the unbroken  $SO(2)$  subgroup of the global  $SO(5)$  symmetry.

Next we want to show that this non-trivial instanton structure is consistent at the surface of a  $3 + 1D$  boson SPT with  $SO(5)$  symmetry. In other words we can regularize  $QCD_3$  with its full  $SO(5)$  symmetry as an on-site symmetry at the boundary of a  $3 + 1D$  bosonic SPT phase.

First, let us discuss possible  $SO(5)$  symmetric boson SPTs in  $3 + 1D$ . Consider any short range entangled gapped phase of an  $SO(5)$  symmetric boson theory, and again couple in background  $SO(5)$  gauge fields. The bulk again admits  $Z_2$ -indexed monopoles in this gauge field, which can be chosen to break  $SO(5)$  to  $SO(2) \times SO(3)$ .

Now there are logically two sharply distinct possibilities: does the monopole transform as a spinor under the unbroken  $SO(3)$  symmetry or not? If it is a spinor then the original gapped state is an SPT state. The other question we may ask is what the charge is under the unbroken  $SO(2)$  symmetry. This charge can be continuously tuned by changing the  $SO(5)$   $\theta$ -angle:

$$\mathcal{L} = \frac{\theta}{4(2\pi)^2} \text{Tr}_{SO(5)}(\mathcal{F}^5 \wedge \mathcal{F}^5) \quad (109)$$

<sup>24</sup> A formal but elegant formulation of this consideration is described in section VII in terms of matching the second Stiefel-Whitney classes of the bundles corresponding to the background and dynamical gauge fields.

where  $\mathcal{F}_{\mu\nu}^5 = \partial_\mu \mathcal{A}_\nu^5 - \partial_\nu \mathcal{A}_\mu^5 - i[\mathcal{A}_\mu^5, \mathcal{A}_\nu^5]$  is the  $SO(5)$  field strength. Since  $\theta$  is a continuous parameter, in the absence of additional symmetries (e.g. time-reversal) it does not label a distinct SPT phase. It is crucial to note that changing  $\theta$  does not affect the  $SO(3)$  transformation properties of the monopole. Thus, the  $SO(5)$  SPT where the  $SO(2)$  monopole is an  $SO(3)$  spinor is an SPT rather distinct from the more familiar boson and fermion topological insulators protected by  $U(1)$  and time-reversal symmetries. Below, we will discuss the topological action for this SPT. Finally, we may ask whether the monopole is a boson or fermion.<sup>25</sup> This property may be altered by shifting  $\theta \rightarrow \theta + 2\pi$  [70, 71].

Now, let us assume the monopole is a boson that carries no  $SO(2)$  charge. Then if it does not transform as a spinor under  $SO(3)$ , the original gapped state is totally trivial. If it is a spinor under  $SO(3)$  then the original gapped state is an SPT state which has the exact same monopole structure to compensate for the instanton structure of  $QCD_3$  with  $N_f = 2$  as a potential boundary state.

#### D. Explicit constructions for the $SO(5)$ SPT

First let us argue that such a 3+1D SPT state indeed exists in a system of  $SO(5)$  symmetric bosons by a coupled layer construction. Notice that though  $N_f = 2$   $QCD_3$  has an anomalous  $SO(5)$  symmetry, the anomaly disappears if we take two copies of it.<sup>26</sup> This is because the  $SO(5)$  monopole (instanton) then gets a spin-1/2 (under  $SO(3)$ ) from each copy and hence can always be made trivial.

We can now construct the required 3 + 1D bosonic SPT state by starting with a stack of 2D layers, each containing 2 copies of  $QCD_3$ . We take one copy from one layer and trivialise it by pairing with another copy from the next layer. This will give a trivial gapped bulk, but at the boundary we are left with single copy of  $QCD_3$ .

We can also construct the bulk boson SPT more explicitly using fermionic partons, following a similar approach to Refs [72, 73]. Consider first a 3 + 1D fermionic topological superconductor with  $SO(8) \times Z_2^T$  symmetry. A continuum model for this state consists simply of 8 relativistic, free,

<sup>25</sup> For simplicity, we may define monopole statistics by making just the  $SO(2)$  part of the gauge group dynamical.

<sup>26</sup> For instance, if  $X_{1,2}$  are  $X$  fields from the two copies of Eq. (100), we can add an  $SO(5)$  symmetric mass term  $mTr(\bar{X}_1 X_1 - \bar{X}_2 X_2)$ . Integrating  $X_{1,2}$  out, we get a pure Yang-Mills theory for  $a$  with no Chern-Simons term, which is expected to confine.

massive Majorana fermions,

$$\mathcal{L} = \sum_{i=1}^8 \bar{\chi}_i (i\gamma^\mu \partial_\mu + m) \chi_i \quad (110)$$

with  $\bar{\chi}_i = \chi_i^T \gamma^0$ . For one sign of the Majorana mass  $m$  we will have a trivial gapped state, while for the other sign we will have a topological superconductor. The 2 + 1D surface of this free-fermion state correspondingly has 8 massless Majorana cones. Now let us couple this system to a dynamical SU(2) gauge field  $a$ . As in our 2+1D discussion, we label the 8 Majoranas  $\chi$  by indices  $m = 0, 1, 2, 3$ ;  $v = 1, 2$  and form the field  $X_{\alpha,v;\beta}$  in Eq. (97). The SU(2) gauge symmetry acts on  $X$  from the right, as in Eq. (86), and the 3 + 1D gauge action has the form,

$$\mathcal{L}_{QCD_4} = \text{tr}(\bar{X}(i\gamma^\mu D_\mu^a + m)X). \quad (111)$$

In the bulk this gauge theory describes a bosonic system with  $\text{SO}(5) \times Z_2^T$  symmetry.  $\text{SO}(5)$  is realized projectively on the Majorana fermions, which form an  $\text{Sp}(4)$  fundamental (101). As in 2 + 1D (Eq. 103), we can rewrite Eq. (111) as two flavors of Dirac fermions with the same mass  $m$  coupled to an SU(2) gauge field.

What state does the theory (111) realize? First consider this theory on a closed manifold. Then integrating out the massive fermions produces, at long wavelengths, the standard Yang-Mills action for the dynamical SU(2) gauge field with no topological term. Indeed, each flavor of Dirac fermions with inverted mass would give rise to an SU(2)  $\theta$ -term in the action with  $\theta = \pi$ ,

$$\mathcal{L}_\theta = \frac{\theta}{2(2\pi)^2} \text{tr}_{\text{SU}(2)} f \wedge f. \quad (112)$$

Thus, two flavors of Dirac fermions with the same mass give  $\theta = 2\pi$ , which is equivalent to  $\theta = 0$  (see section VII for a more careful discussion). It is expected that the pure Yang-Mills theory will confine, and so the ground state is seemingly trivial. Now consider the theory in the presence of a boundary. Though the bulk is confined, the boundary is precisely the QCD<sub>3</sub> theory of interest to us with global  $\text{SO}(5) \times Z_2^T$  symmetry. As promised, the bulk theory therefore describes the SPT phase of bosons with  $\text{SO}(5) \times Z_2^T$  symmetry. In principle this construction could also be used to write a variational (Gutzwiller-projected) wavefunction for a lattice SO(5) topological paramagnet.<sup>27</sup>

<sup>27</sup> Though this parton construction made use of a  $Z_2^T$  invariant fermion SPT, the final boson SPT is stable even if  $Z_2^T$  is broken. As we discuss below, with full  $\text{SO}(5) \times Z_2^T$  symmetry the response to a background  $\text{SO}(5)$  gauge field has a discrete  $\theta$  term, but not the conventional  $\theta$  term of Eqn. 109. If now  $Z_2^T$  is broken but  $\text{SO}(5)$  is preserved, then such a conventional theta term is allowed. Regardless, the presence of the discrete theta term means that we still have a non-trivial SPT phase. More physically the  $\text{SO}(5)$  monopole which breaks  $\text{SO}(5)$  to  $\text{SO}(2) \times \text{SO}(3)$  will still transform as a spinor under  $\text{SO}(3)$  but in the absence of  $Z_2^T$  is allowed to have non-zero  $\text{SO}(2)$  charge.

How do we formally describe the bulk response to a background  $\text{SO}(5)$  gauge field that captures the structure of the  $Z_2$ -indexed monopole in these systems? In section VII we show that the partition function of the bulk SPT phase takes the form

$$Z[\mathcal{A}^5] = |Z[\mathcal{A}^5]| e^{i\pi \int w_4[\mathcal{A}^5]}. \quad (113)$$

Here  $w_4[\mathcal{A}^5] = 0, 1$  is a quantity known as the fourth Steifel-Whitney class of the  $\text{SO}(5)$  gauge bundle  $\mathcal{A}^5$ .<sup>28</sup> The phase  $e^{i\pi \int w_4[\mathcal{A}^5]}$  is the analog of the familiar  $\theta$ -term response of the standard topological insulator to background  $U(1)$  gauge fields. In contrast to the usual case here the  $\theta$ -angle is restricted to two discrete values: 0 (corresponding to a totally trivial state) or  $\pi$  (corresponding to the SPT phase of interest to us here). Precisely such a discrete  $\theta$  term was introduced a few years ago in Ref. [42] for non-abelian gauge theories. In that work the possibility of such  $\theta$  terms was pointed out and some of their physical consequences were discussed. We see that such  $\theta$  terms emerge naturally in the response of bosonic SPT phases. Our discussion above can be viewed as a construction of an  $\text{SO}(5)$  symmetric bosonic 3 + 1D model whose response includes these discrete  $\theta$  terms. Indeed, in section VII, we show explicitly that the theory (111) has this discrete  $\theta$ -term in its response to a background  $\text{SO}(5)$  gauge field.

### E. Symmetry-enforced gaplessness

As mentioned above the IR fate of  $N_f = 2$  QCD<sub>3</sub> is at present unclear. However our understanding of the anomalous symmetry realization in this theory enables us to derive some general restrictions. We will show that either the  $\text{SO}(5) \times Z_2^T$  symmetry is spontaneously broken or the theory is gapless in the IR. This result follows purely from the anomalous symmetry realization. Indeed, it is a general feature of the surface of the bulk 3 + 1D boson SPT with  $\text{SO}(5) \times Z_2^T$  symmetry discussed in the previous subsection. Such a phenomenon was first described for some fermion SPTs in Ref. [74] and dubbed ‘symmetry enforced gaplessness’. Other examples, including some boson SPTs, are described in Refs. [75, 76].

Consider a putative gapped state of the 2 + 1D theory that preserves the  $\text{SO}(5) \times Z_2^T$  symmetry. All the quasiparticles of this state must transform under some representation, possibly projective. As usual if a quasiparticle transforms non-projectively under  $\text{SO}(5)$  we can ‘screen’ it

<sup>28</sup> Strictly speaking,  $\mathcal{A}^5$  is the connection on the  $\text{SO}(5)$  gauge bundle. Here, we abuse the notation a bit by using  $\mathcal{A}^5$  to label both the bundle and the connection.

using composites of the  $\hat{n}$ -vector to make it a singlet under  $SO(5)$ . Therefore the only non-trivial symmetry possibility is a quasiparticle transforming under the 4-dimensional spinor representation (fundamental of  $Sp(4)$ ). Let us call such quasiparticles  $X_I$ .

We will think of  $QCD_3$  as living on the surface of the  $3 + 1D$  SPT described above. Now let us tunnel in a  $Z_2$ -valued  $SO(5)$  monopole through the surface. We know that the monopole breaks  $SO(5)$  to  $SO(2) \times SO(3)$ , and that it transforms in the  $(0, 1/2)$  representation of  $SO(2) \times SO(3)$ . Therefore, in order for  $SO(2) \times SO(3)$  charge to be conserved, there must be a quasiparticle in this putative gapped surface state with these properties. However the only quasiparticles which transform non-trivially under  $SO(5)$  are the “spinors”  $X_I$ . They transform with  $SO(2)$  charge of  $\frac{1}{2}$  and as a spinor under the  $SO(3)$ , and not under the  $(0, 1/2)$  representation. It follows that the gapped state we imagined cannot have the right anomaly, and hence is not a possible surface state.

Note, however, that if time reversal is broken then there can be a Hall conductivity for the  $SO(5)$  currents. Then the  $SO(5)$  monopole threading will nucleate an  $SO(2)$  charge determined by the Hall conductivity. This can then combine with the  $SO(5)$  spinor  $X_I$  to produce an object with  $(0, 1/2)$  quantum numbers under the  $SO(2) \times SO(3)$  symmetry, as required. Thus, if time reversal is broken, a gapped  $SO(5)$  symmetry preserving state is no longer prohibited. Indeed it is easy to construct a ‘chiral spin liquid’ state explicitly: see section VII D.

The conclusion therefore is that, in the IR,  $N_f = 2$   $QCD_3$  with  $SO(5) \times Z_2^T$  symmetry must either spontaneously break the symmetry or be gapless, *i.e* flow to a CFT. It cannot be fully gapped while preserving symmetries even if we allow for non-trivial topological order.

As another application of this result, consider the fate of the Neel-VBS transition at the longest distances. One interpretation of the existing numerics is to say that the RG flows are attracted to a ray with  $SO(5) \times Z_2^T$  symmetry. If so then the eventual destination of this ray is either a weak first order transition, a  $Z_2^T$  broken spin liquid, or a gapless CFT — a gapped, symmetry-preserving, topologically ordered state is ruled out.<sup>29</sup>

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<sup>29</sup> Of course if the ray ends at a gapless CFT where allowed  $SO(5)$ -breaking terms are relevant, these terms will eventually become large and the above constraints will not apply.

## VII. QCD<sub>3</sub> AS THE SURFACE OF AN $SO(5)$ INVARIANT 3 + 1D SPT: FORMAL DESCRIPTION

Here we expand on the discussion in section VI C and demonstrate more formally that  $SO(5)$  symmetric  $N_f = 2$  QCD<sub>3</sub>, Eq. 100, can be interpreted as a surface theory of a bosonic  $SO(5)$ -protected 3 + 1D SPT phase. We will first develop a precise formal description of the anomaly of QCD<sub>3</sub> and show how it is compensated by the 3 + 1D SPT bulk. We will also sharpen the parton construction of this SPT outlined in Sec. VI C to derive the bulk partition function in the presence of a background  $SO(5)$  gauge field. We will explicitly derive the advertised discrete theta term. The reader is referred to Appendix D for a brief summary of math concepts that will be useful in this section and to Appendix E for a review of the field-theoretic description of topological superconductors.

### A. Gauging $SO(3)_s$

First, it is useful to recall how the gauge field associated with the  $SO(3)_s$  spin-rotation symmetry enters in QCD<sub>3</sub>. Since  $SO(3)_s$  symmetry is a true on-site symmetry of the lattice spin system, its implementation in QCD<sub>3</sub> must be non-anomalous. Given an  $SO(3)_s$  gauge bundle  $E_s$  over our space-time 3 manifold  $M$ ,<sup>30</sup> we attempt to lift it to an  $SU(2)_s$  gauge bundle  $\widehat{E}_s$ . The resulting  $SU(2)_s$  transition functions  $\widehat{U}_{ij}^s$  might fail to satisfy the cocycle condition:  $\widehat{U}_{ik}^s \widehat{U}_{kj}^s \widehat{U}_{ji}^s = (-1)^{w_2(E_s)_{kji}}$ , where  $w_2(E_s) \in H^2(M, Z_2)$  is a representative of the second Stiefel-Whitney class of  $E_s$  [77]. In this case, we take the transition functions of our dynamical gauge field  $SU(2)_g$  to violate the cocycle condition by precisely the same factor in the center of  $SU(2)_g$ . That is, if we project the  $SU(2)_g$  transition functions  $\widehat{U}^g$  to  $SO(3)_g$ , we get an  $SO(3)$  bundle  $E_g$  with  $w_2(E_g) = w_2(E_s)$ . The fermions  $\chi_{m,v}$  then see an  $SO(4) = (SU(2)_s \times SU(2)_g)/Z_2$  gauge field which satisfies the cocycle condition. In the background of this  $SO(4)$  gauge field  $A^{s,g}$ , the action (100) becomes,

$$L = i\bar{\chi}_v \gamma^\mu (\partial_\mu - iA^{s,g}) \chi_v \quad (114)$$

where  $A^{s,g}$  lives in  $so(4)$  Lie algebra and acts on the spin-color index  $m$  of  $\chi_{m,v}$  but not on the valley index  $v$ . We note that in addition to the  $SO(4)$  gauge field, the fermions  $\chi_{m,v}$  also see a spin

<sup>30</sup> As usual, we transition to Euclidean space-time here. Furthermore, we will formulate the theory on an arbitrary space-time manifold  $M$ . The ability to do so is a useful formal consistency check on the theory. We assume that the manifold  $M$  is oriented: i.e. unless otherwise noted, we will not discuss “gauging” of time-reversal symmetry.

structure (we have suppressed the spin-connection above) with transition functions  $h_{ij} \in Spin(3)$ . Every three-manifold is spin, so  $h_{ij}$  can always be chosen to satisfy the cocycle condition (as we assumed in the discussion above). However, it will be more useful to think of  $h_{ij}$  as arbitrary lifts to  $Spin(3)$  of the  $SO(3)$  transition functions in the tangent bundle. The fermions only see a combination of  $SU(2)_s$ ,  $SU(2)_g$  and  $Spin(3)_{TM}$  transition functions, so we only require this combination to satisfy the cocycle condition.<sup>31</sup> A change  $h_{ij} \rightarrow \zeta_{ij} h_{ij}$ ,  $\zeta_{ij} = \pm 1$ , can be compensated by modifying the  $SU(2)_g$  bundle,  $\widehat{U}_{ij}^g \rightarrow \zeta_{ij} \widehat{U}_{ij}^g$ , so the theory does not depend on the spin structure. Further,  $w_2(E_s) + w_2(E_g) + w_2(TM) = 0 \pmod{2}$ , where we are now thinking of  $w_2$ 's as concrete cochains representing  $H^2(M, \mathbb{Z}_2)$  (as already noted, as a cohomology class,  $w_2(TM) = 0$ ).

We now discuss the regularization of Eq. (114) - in principle, such regularization is provided by the lattice we started with. An equivalent continuum regularization is obtained by using Pauli-Villars (PV) regulators with opposite mass for the Majoranas in the two valleys  $v = 1, 2$ . Recall that for a single  $SO(n)$  vector Majorana fermion coupled to an  $SO(n)$  gauge field  $A$ , the PV regulated partition function is given by,<sup>[54]</sup>

$$Z_{\chi;PV,\pm}(A) = |Z_\chi(A)| \exp(\mp \pi i \eta(iD_A)/4) \quad (115)$$

where the sign in the exponent is determined by the sign of Pauli-Villars mass. Here,  $\eta(iD_A)$  is the  $\eta$ -invariant of the Dirac operator  $iD_A = i\gamma^\mu(\partial_\mu + i\omega_\mu - iA_\mu)$ ,<sup>32</sup>

$$\begin{aligned} \eta &= \eta(0) + N_0 \\ \eta(s) &= \sum_{\lambda \neq 0} \text{sgn}(\lambda) |\lambda|^{-s} \end{aligned} \quad (116)$$

with  $\lambda$  in the above sum - eigenvalues of  $iD_A$ , and  $N_0$  - the number of zero modes of  $iD_A$ . So, when we use PV regulators of opposite mass for the two valleys  $v = 1, 2$ , we obtain after integrating the Majorana fermions out

$$\exp(-S_{QCD_3}[A^{s,g}]) = |Z_{\chi_{v=1}}(A^{s,g})|^2 \quad (117)$$

with  $Z_{\chi_{v=1}}(A^{s,g})$  — the partition function of Majoranas in just a single valley  $v = 1$ . The theory thus defined obviously preserves  $SO(3)_s$  as a non-anomalous symmetry. Likewise, time-reversal symmetry (last line of (96)) is preserved and non-anomalous — this must be the case, as it is an

<sup>31</sup> The subscript  $TM$  on  $Spin(3)$  reminds us that these are transition functions associated with the tangent bundle.

<sup>32</sup> Here,  $\omega_\mu$  is the spin connection, which we included for further generality to describe curved manifolds [78].



onsite symmetry of the initial lattice model.<sup>33</sup> The discrete lattice symmetries in (96) are global symmetries of (117), however, they are (in a certain sense) anomalous: there is no contradiction here, since they are not realized by the original lattice model in an on-site manner.

For future reference, we note that we can obtain an equivalent theory (117) by regulating both valleys in the same way (with the same sign of PV mass) and supplementing the action by a Chern-Simons term for  $A^{s,g}$ :

$$S_{QCD_3} = \int_M [\bar{\chi}_v \gamma^\mu (\partial_\mu + i\omega_\mu - iA^{s,g}) \chi_v]_{PV,+} - iCS_{SO(4)}[A^{s,g}, Y_4] - 4iCS_g[Y_4] \quad (118)$$

where the subscript  $PV_+$  indicates that the PV mass is the same for both valleys, see Eq. (115). We will use the notation where  $CS_{SO(n)}[A, Y_4]$  is the Chern-Simons action for  $SO(n)$  gauge field  $A$  at level 1,<sup>34</sup> and  $CS_g[Y_4]$  is the gravitational Chern-Simons action (corresponding to the gravitational response of a  $p_x + ip_y$  superconductor). The significance of the parameter  $Y_4$  is as follows. We recall that a technical trick to define a Chern-Simons term is to extend the three-manifold  $M$  to a four-manifold  $Y_4$ , and also extend the gauge field  $A$  from  $M$  to  $Y_4$ :

$$\begin{aligned} CS_{SO(n)}[A, Y_4] &= \frac{\pi}{2 \cdot (2\pi)^2} \int_{Y_4} \text{tr}_{SO(n)} F \wedge F \\ CS_g[Y_4] &= \frac{\pi}{8 \cdot 24\pi^2} \int_{Y_4} \text{tr} R \wedge R \end{aligned} \quad (119)$$

where  $F = dA - iA \wedge A$  is the  $SO(n)$  field strength,  $R$  is the Riemann curvature tensor, and the trace  $\text{tr}_{SO(n)}$  is in the  $n$ -dimensional vector representation. In order for a theory to be a well-defined strictly 2+1D theory, it must be independent of the choice of the four-manifold  $Y_4$  and the particular extension of the gauge field to  $Y_4$ . For our theory (118) this is actually guaranteed by the Atiyah-Patodi-Singer (APS) theorem. Indeed, recall that by APS theorem, if our three manifold  $M$  is endowed with an  $(SO(n)_A \times Spin(3)_{TM})/Z_2$  bundle  $E$  (where  $n$  is even) and  $M$  is the boundary of a four-manifold  $Y_4$  such that  $E$  extends to a  $(SO(n)_A \times Spin(4)_{TY_4})/Z_2$  bundle over  $Y_4$  then [78]

$$\frac{\pi}{2} \eta(iD_{A,M}^{SO(n)}) = CS_{SO(n)}[A, Y_4] + nCS_g[Y_4] - 2\pi \mathcal{J}[A, Y_4] \quad (120)$$

<sup>33</sup> Strictly speaking, to show that  $T$  is non-anomalous we have to regulate the theory on a non-orientable manifold. However, this can be done as the mass term  $\chi^T \tau^y \mu^z \chi$  preserves  $T$  - we can introduce a PV regulator with this mass term.

<sup>34</sup> Level 1 is defined so that for an  $SO(2)$  subgroup acting on the first two components of an  $n$ -vector,  $\sigma_{xy} = 1$ . In other words, level 1 corresponds to the  $SO(n)$  response of  $n$  identical copies of a  $p_x + ip_y$  superconductor.

where  $2\mathcal{J}[A, Y_4]$  is the index of the Dirac operator  $iD_A$  on  $Y_4$  with APS boundary conditions.<sup>35</sup> Since the left-hand-side of (120) only depends on the boundary data, the sum  $CS_{SO(n)}[A, Y_4] + nCS_g[Y_4]$  is independent of the extension chosen modulo  $2\pi$ .<sup>36</sup> This means that (118) is a well-defined strictly 2+1D theory. Furthermore, integrating the fermions in (118) out using Eq. (115) and applying (120), we recover the original regularization (117).

Given the definition (119) of  $SO(4)$  Chern-Simons term via the extension to  $Y_4$ , we can re-write it in terms of field strength of  $SU(2)_s$  and  $SU(2)_g$  gauge fields  $A^s$  and  $a^g$ , or alternatively their  $SO(3)$  representations,<sup>37</sup> obtaining,

$$CS_{SO(4)}[A^{s,g}, Y_4] + 4CS_g[Y_4] = CS_{SU(2)}[A^s, Y_4] + CS_{SU(2)}[a^g, Y_4] + 4CS_g[Y_4] \quad (121)$$

$$= \frac{1}{2}CS_{SO(3)}[A^s, Y_4] + \frac{1}{2}CS_{SO(3)}[a^g, Y_4] + 4CS_g[Y_4] \quad (122)$$

where as usual for an  $SU(2)$  gauge field  $A$ ,

$$CS_{SU(2)}[A, Y_4] = \frac{1}{4\pi} \int_{Y_4} \text{tr}_{SU(2)} F \wedge F \quad (123)$$

and the trace is in the spin-1/2 representation. The half-integer-level  $SO(3)$  Chern-Simons terms in (122) are not independent of  $Y_4$  individually, but the full sum is independent of  $Y_4$  (the integer-level  $SU(2)$  terms in (121) are also not individually independent of  $Y_4$ , since transition functions for  $SU(2)_s$  and  $SU(2)_g$  don't independently satisfy the cocycle condition). It is instructive to check this statement without appealing to the APS theorem. To show that (122) is independent of  $Y_4$ , it suffices to check that it vanishes modulo  $2\pi$  when  $Y_4$  has no boundary. Recalling that for an  $SO(n)$  gauge bundle on a closed manifold  $Y_4$  the first Pontryagin number is given by [77]

$$p_1 = \frac{1}{2 \cdot (2\pi)^2} \int_{Y_4} \text{tr}_{SO(n)} F \wedge F \quad (124)$$

and the signature of the manifold is [77]

$$\sigma = -\frac{1}{24\pi^2} \int_{Y_4} \text{tr} R \wedge R \quad (125)$$

<sup>35</sup> For an  $SO(n)$  gauge-field the index is always even as  $[CK, iD_A] = 0$ , and  $(CK)^2 = -1$ , with  $C$  - the charge conjugation operator and  $K$  - complex conjugation.

<sup>36</sup> In the case when  $M$  is supplemented with a spin-structure, which is preserved by the extension,  $CS_{SO(n)}[A, Y_4]$  and  $CS_g[Y_4]$  are *separately* independent of the extension.

<sup>37</sup> In this section, we will be slightly sloppy and often use  $A$  to label both the bundle and the connection. Moreover, we will use  $A^s$  to label both the  $SO(3)_s$  bundle and the  $SU(2)_s$  lift. The meaning will be clear from the context. Similarly,  $A^5$  will denote both the  $SO(5)$  bundle and the  $Sp(4)$  lift.

we must show that

$$p_1^{SO(3)}[A^s, Y_4] + p_1^{SO(3)}[a^g, Y_4] - \sigma[Y_4] = 0 \pmod{4} \quad (126)$$

for any closed  $Y_4$ . Now, for an  $SO(n)$  gauge bundle,[42]

$$p_1 = \mathcal{P}(w_2) + 2w_4 \pmod{4} \quad (127)$$

where  $w_i$  are the Stiefel-Whitney classes of the bundle and  $\mathcal{P} : H^2(Z_2) \rightarrow H^4(Z_4)$  is the Pontryagin square operation, which satisfies  $\mathcal{P}(a + b) = \mathcal{P}(a) + \mathcal{P}(b) + 2a \cup b \pmod{4}$  (see appendix D for details). Recalling that  $w_2[A^s] + w_2[a^g] + w_2(TY_4) = 0$ , and that for  $SO(3)$  bundles  $w_4 = 0$ , Eq. (126) reduces to

$$p_1[A^s, Y_4] + p_1[a^g, Y_4] - \sigma[Y_4] = \int_{Y_4} (2\mathcal{P}(w_2[A^s]) + 2w_2[A^s] \cup w_2[TY_4] + \mathcal{P}(w_2[TY_4])) - \sigma[Y_4] \quad (128)$$

where all manipulations are modulo 4. On an orientable four-manifold  $Y_4$ , for any  $a \in H^2(Z_2)$ ,  $a \cup a = a \cup w_2[TY_4]$  (see e.g. Ref. [79], p. 132); furthermore,  $\mathcal{P}(a) = a \cup a \pmod{2}$ , so the first two terms on the RHS of Eq. (128) add to 0 mod 4. The remaining statement  $\int_{Y_4} \mathcal{P}(w_2[TY_4]) = \sigma \pmod{4}$  is also true.<sup>38</sup>

## B. Gauging $SO(5)$

We are now ready to discuss gauging of the full  $SO(5)$  global symmetry of  $QCD_3$ . Given an  $SO(5)$  gauge bundle with connection  $A^5$  on our three-manifold  $M$ , we attempt to lift it to  $Sp(4) = Spin(5)$ . The resulting transition functions may not satisfy the cocycle condition: the defect is  $w_2[A^5]$ . As before, we choose  $SU(2)_g$  transition functions so that the combination of  $Sp(4)$ ,  $SU(2)_g$  and  $Spin(3)_{TM}$  transition functions satisfies the cocycle condition, i.e.  $w_2[A^5] + w_2[a^g] + w_2[TM] = 0 \pmod{2}$ , with  $w_2[a^g]$  being the second Stiefel-Whitney class of the  $SO(3)_g$  gauge bundle. Thinking

<sup>38</sup> There is a theorem that on any closed four-manifold,  $w_2(TY_4)$  can be lifted to an integer cohomology class  $\hat{w}_2 \in H^2(Z)$ . (This is the statement that every four manifold admits a  $Spin_c$  structure.) For an integer cohomology class  $a$ ,  $\mathcal{P}(a) = a \cup a$ . So we need to show  $\int_{Y_4} \hat{w}_2 \cup \hat{w}_2 = \sigma \pmod{4}$ . Letting the unimodular matrix  $Q$  be the intersection pairing on  $\text{Free}(H^2(Y_4, Z))$ , we must show  $(\hat{w}_2)^T Q \hat{w}_2 = \sigma(Q) \pmod{4}$ , where  $\sigma(Q)$  is the signature of the matrix  $Q$ . Actually, a stronger statement  $(\hat{w}_2)^T Q \hat{w}_2 = \sigma(Q) \pmod{8}$  holds and can be derived purely from algebra (see Ref. [80], p. 24). Actually, this statement is familiar to users of the APS theorem. Consider a general  $Spin_c$  connection with field strength  $F$  on a four manifold. The Atiyah-Singer theorem [77, 78] then tells us that  $\frac{1}{2} \int \frac{F}{2\pi} \wedge \frac{F}{2\pi} - \frac{\sigma}{8} \in Z$ . Now a  $Spin_c$  connection has  $\int \frac{F}{2\pi} = \int \frac{\hat{w}_2}{2} \pmod{Z}$  for any oriented two-cycle. Choosing a special case where  $\frac{F}{2\pi} = \frac{\hat{w}_2}{2}$ , we obtain the needed result.

of  $(Sp(4) \times SU(2)_g)/Z_2$  as a subgroup of  $SO(8)$ , the (unregulated) action becomes

$$L_{QCD_3} = \bar{\chi} \gamma^\mu (\partial_\mu + i\omega_\mu - iA_\mu^{5,g}) \chi \quad (129)$$

with  $A^{5,g}$  living in the  $so(8)$  Lie algebra, i.e. acting on spin-color  $m$  and valley indices  $v$  of  $\chi$ . We must now specify how to regulate the above action. We can no longer use PV regulators of opposite mass for the two valleys since this will break  $SO(5)$  symmetry. Instead, we use a common PV regulator for the  $SO(8)$  vector  $\chi$  and supplement the action by an  $SO(8)$  Chern-Simons term,

$$S_{QCD_3} = \int_M (\bar{\chi} \gamma^\mu (\partial_\mu + i\omega_\mu - iA_\mu^{5,g}) \chi)_{PV,+} - \frac{i}{2} CS_{SO(8)}[A^{5,g}, Y_4] - 4iCS_g[Y_4] \quad (130)$$

with the Chern-Simons terms again defined by extending to a four-manifold  $Y_4$ , as before. When the  $SO(5)$  bundle reduces to an  $SO(3)_s$  bundle, Eq. (130) reduces to Eq. (118) as needed. But are the Chern-Simons terms in Eq. (130) independent of  $Y_4$  for an arbitrary  $SO(5)$  bundle? We will show that the answer is no: Eq. (130) is not well-defined as a purely 2 + 1D theory. However, we will be able to define it as a surface of an  $SO(5)$  protected 3 + 1D bosonic SPT phase.

First, we observe that if  $A^{5,g}$  was an arbitrary  $SO(8)$  bundle, (130) obviously would not define a purely 2 + 1D theory, as the  $SO(8)$  level is fractional. In fact, physically Eq. (130) is just the action of eight identical copies of a 3 + 1D topological superconductor living on the space  $Y_4$  and coupled to an  $SO(8)$  gauge field. The bulk of such a state has a non-trivial  $SO(8)$  response. Indeed, by the APS theorem (120), the partition function of (130) after integrating the fermions out becomes,

$$\exp(-S_{QCD_3}[A^{5,g}]) = |Z_\chi[A^{5,g}]|(-1)^{\mathcal{J}[A^{5,g}, Y_4]} \quad (131)$$

with  $|Z_\chi[A^{5,g}]|$  — the absolute value of the partition function for our eight 2+1D Majorana fermions coupled to  $A^{5,g}$ , and  $2\mathcal{J}$  — the index of  $iD_{A^{5,g}}$  on  $Y_4$ . For closed  $Y_4$  and a general  $SO(8)$  gauge field,  $\mathcal{J}$  is not necessarily even, so  $S_{QCD_3}[A^{5,g}]$  depends on the extension to  $Y_4$ . However, our  $A^{5,g}$  is not the most general  $SO(8)$  gauge field, rather we are dealing with an  $(Sp(4) \times SU(2)_g \times Spin(3)_{TM})/(Z_2 \times Z_2)$  bundle: when this bundle is extended to the four-manifold  $Y_4$ , is Eq. (130) independent of the extension?<sup>39</sup> We rewrite,

$$\frac{1}{2}CS_{SO(8)}[A^{5,g}, Y_4] = \frac{1}{2}CS_{SO(5)}[A^5, Y_4] + \frac{1}{2}CS_{SO(3)}[a^g, Y_4] \quad (132)$$

<sup>39</sup> There is also one caveat here: does an extension always exist? For this, one must calculate the 3D bordism group — a task that we will not attempt here.

Thus, to check whether (130) is well-defined as a 2+1D theory, we must determine whether for closed  $Y_4$ ,

$$p_1^{SO(5)}[A^5, Y_4] + p_1^{SO(3)}[a^g, Y_4] - \sigma[Y_4] \stackrel{?}{=} 0 \pmod{4} \quad (133)$$

We again use the identity (127), and  $w_2[A^5] + w_2[a^g] + w_2[TM] = 0$ . Repeating the manipulations below (126), we obtain

$$p_1[A^5, Y_4] + p_1[a^g, Y_4] - \sigma[Y_4] = 2w_4[A^5, Y_4] \neq 0 \pmod{4} \quad (134)$$

Thus, Eq. (130) generally depends on the extension to  $Y_4$ : given two extensions to  $Y_4$  and  $\tilde{Y}_4$ , we have,

$$\exp\left(-S_{QCD_3}[A^{5,g}, \tilde{Y}_4] + S_{QCD_3}[A^{5,g}, Y_4]\right) = \exp\left(\pi i \int_{\tilde{Y}_4 \cup \tilde{Y}_4} w_4[A^5]\right) \quad (135)$$

where the last integral is over the manifold obtained by gluing together  $\tilde{Y}_4$  and  $Y_4$  with reversed orientation. Crucially, the variation (135) only depends on the extension of the  $SO(5)$  bundle  $A^5$ , but not on the  $SU(2)_g$  bundle. Still, we cannot promote  $SO(5)$  to an on-site symmetry of a strictly 2 + 1D theory. However, we can think of the theory (130) as the surface of an  $SO(5)$  protected 3 + 1D SPT, as follows. Let  $X_4$  be the physical 3 + 1D manifold that our SPT phase lives on. There is an  $SO(5)$  gauge field  $A^5$  on  $X_4$ . When  $X_4$  has no boundary, we let the partition function be:

$$Z_{3+1}[A^5, X_4] = \exp\left(\pi i \int_{X_4} w_4[A^5]\right) \quad (136)$$

When  $X_4$  has a boundary,  $\partial X_4 = M$ , Eq. (136) is not well-defined (not gauge-invariant under using different representatives of the cohomology class  $w_4[A^5] \in H^4(X_4, \mathbb{Z}_2)$ ). However, we can combine the surface action (130) with the bulk action (136), to obtain

$$\begin{aligned} S_{QCD_3}^{\text{bulk}+\text{bound}} &= \int_M [\bar{\chi}_v \gamma^\mu (\partial_\mu + i\omega_\mu - iA^{5,g}) \chi_v]_{PV,+} \\ &\quad - \frac{i}{2} CS_{SO(5)}[A^5, Y_4] - \frac{i}{2} CS_{SO(3)}[a^g, Y_4] - 4i CS_g[Y_4] \\ &\quad + \pi i \int_{X_4 \cup \tilde{Y}_4} w_4[A^5] \end{aligned} \quad (137)$$

Again, the Chern-Simons terms in the second line of (137) are defined with the help of an extension to an auxiliary four-dimensional manifold  $Y_4$ . The term in the last line of Eq. (137) involves an integral over a manifold obtained by gluing  $X_4$  and  $Y_4$  with its orientation reversed. Every line in Eq. (137) is well-defined, however, the second and third lines individually depend on the extension to  $Y_4$ . However, the  $Y_4$  dependence of the second and third lines cancels, due to Eq. (135). Therefore,

Eq. (137) is overall independent of  $Y_4$ . It does, however, depend on  $A^5$  on the physical 3 + 1D manifold  $X_4$  and reduces to Eq. (136) when  $X_4$  has no boundary. Therefore, Eq. (137) is a perfectly well defined action of an  $SO(5)$  protected SPT phase on the 3 + 1D manifold  $X_4$  with boundary  $M$ : only the background  $SO(5)$  gauge-field  $A^5$  lives in the bulk; the fields  $a^g$ ,  $\chi$  live on the surface. The independence of the variation (135) of  $a^g$  was crucial for such an SPT interpretation to be possible.

Let us obtain some intuition for the bulk topological term (136). First, consider gauging just the  $SO(3)_s \times SO(2)_{VBS}$  subgroup of  $SO(5)$ , i.e. the  $SO(5)$  bundle is a direct sum of  $SO(3)_s$  and  $SO(2)_{VBS}$  bundles. Label the corresponding gauge bundles as  $A^s$  and  $A^{VBS}$ . We have

$$\begin{aligned} 2w_4^{SO(5)}[A^5] &= p_1[A^5] - \mathcal{P}(w_2[A^5]) = p_1^{SO(3)}[A^s] + p_1^{SO(2)}[A^{VBS}] - \mathcal{P}(w_2[A^s] + w_2[A^{VBS}]) \\ &= -2w_2[A^s] \cup w_2[A^{VBS}] \pmod{4} \end{aligned} \quad (138)$$

where we've repeatedly used Eq. (127), together with the fact that  $w_2$  and  $p_1$  are additive under  $SO(n)$  bundle addition, and that  $w_4 = 0$  for  $SO(n)$  with  $n \leq 3$ . So,

$$w_4[A^5] = w_2[A^s] \cup w_2[A^{VBS}] = w_2[A^s] \cup \frac{F^{VBS}}{2\pi} \quad (139)$$

where we've used the fact that for an  $SO(2)$  gauge field,  $w_2$  and the first Chern class  $\frac{F}{2\pi}$  coincide mod 2:  $w_2 = \frac{F}{2\pi} \pmod{2}$ .<sup>40</sup> What is the physical interpretation of Eq. (139)? Imagine we take our bulk manifold to be  $S^2 \times \Sigma$ , with  $\Sigma$  an arbitrary two-dimensional surface. Place flux  $2\pi$  of  $F^{VBS}$  through  $S^2$ . Then the partition function (136) is  $Z = \exp(\pi i \int_{\Sigma} w_2[A^s])$ . This is precisely the partition function of  $SO(3)_s$  protected 1 + 1D Haldane phase on  $\Sigma$ .<sup>41</sup> We know that the Haldane phase on a spatial interval  $I = [0, 1]$  has dangling spin 1/2s at the boundary. So, we can guess

<sup>40</sup> Below, we will be somewhat cavalier using  $\frac{F}{2\pi}$  to denote both the Chern class (in  $H^2(Z)$ ) and the field-strength  $F = dA$  of a  $U(1)$  bundle.

<sup>41</sup> One way to obtain this result is the following. Recall that the Haldane chain is described by the  $O(3)$  non-linear sigma model at  $\theta = 2\pi$ . In a  $CP^1$  description the Lagrangian is [81]  $\mathcal{L} = |(\partial_\mu - ia_\mu)z|^2 + i\theta f/2\pi$ , where  $f$  is the field strength associated with  $a_\mu = -iz^\dagger \partial_\mu z$ . A background  $SO(3)_s$  gauge field translates into a background  $SU(2)_s$  gauge field for the  $z$ -field. If, however, the  $SO(3)_s$  gauge bundle has  $w_2 \neq 0$ , we cannot then naively lift it to  $SU(2)_s$ . However,  $z$  is also coupled to the  $U(1)$  gauge field  $a$ . Any defect in the cocycle condition of the  $SO(3)_s$  bundle can be compensated by a  $\pi$  flux of the  $U(1)$  gauge field. Thus, a configuration of the  $SO(3)_s$  gauge field with a non-zero  $w_2$  necessarily induces a background  $a_\mu$  configuration where  $\int_{\Sigma} \frac{f}{2\pi} = \int_{\Sigma} \frac{w_2[A^s]}{2} \pmod{Z}$ . For the Haldane chain, from the  $CP^1$  Lagrangian with the  $\theta$ -term at  $\theta = 2\pi$  we immediately see that the partition function has an extra phase  $e^{i\pi \int_{\Sigma} w_2[A^s]}$ . Clearly on a closed manifold this is invariant under 'gauge transformations',  $w_2 \rightarrow w_2 + dn$ . However, in the presence of a boundary this is no longer true. Gauge invariance can be mended by including a Wilson line  $W[A^s]$  in the spin-1/2 representation along the boundary, i.e. the partition function now becomes  $W[A^s](-1)^{\int_{\Sigma} w_2}$ . This boundary Wilson line is precisely the well known dangling spin-1/2 moment at the boundary of the Haldane chain.

that if we consider the theory on the spatial manifold  $S^2 \times I$ , then there are dangling spin 1/2s at the two ends of  $I$  - i.e. at the locations of  $SO(2)_{VBS}$  monopoles. We, therefore, conclude that the topological term (139) makes monopoles of  $SO(2)_{VBS}$  transform in the spin-1/2 representation of  $SO(3)_s$ . This is precisely the conclusion reached by less formal methods in section VIC.

We can further focus on just the easy-plane subgroup  $SO(2)_s$  of  $SO(3)_s$ , then Eq. (139) reduces to

$$S = \pi i \int_{X_4} w_4[A^5] = \pi i \int_{X_4} \frac{F^s}{2\pi} \cup \frac{F^{VBS}}{2\pi} = \pi i \int_{X_4} \frac{F^s}{2\pi} \wedge \frac{F^{VBS}}{2\pi} \quad (140)$$

This is precisely the mutual  $\theta = \pi$  term for  $U(1)_s \times U(1)_{VBS}$ . Note that to protect this  $\theta$  value from shifting, one needs to rely on some discrete symmetry, such as spin-flip symmetry or time-reversal.

What if we restrict ourselves to an  $SO(4)$  subgroup of  $SO(5)$ ? Let us write  $SO(4) = (SU(2)_L \times SU(2)_R)/Z_2$ . Label associated  $SO(3)_L$  and  $SO(3)_R$  gauge fields as  $A^L$  and  $A^R$ . We have  $w_2[A^L] = w_2[A^R] = w_2[A^5]$ . Now,

$$\begin{aligned} 2w_4[A^5] &= p_1[A^5] - \mathcal{P}(w_2[A^5]) = \frac{1}{2}p_1^{SO(3)}[A^L] + \frac{1}{2}p_1^{SO(3)}[A^R] - \mathcal{P}(w_2[A^R]) \\ &= \frac{1}{2}p_1[A^L] - \frac{1}{2}p_1[A^R] \pmod{4} \end{aligned} \quad (141)$$

Therefore, Eq. (136) reduces to

$$S = \frac{1}{4}CS_{SO(3)}[A^L, X_4] - \frac{1}{4}CS_{SO(3)}[A^R, X_4] = \frac{1}{2}CS_{SU(2)}[A^L, X_4] - \frac{1}{2}CS_{SU(2)}[A^R, X_4] \quad (142)$$

In  $SU(2)$  terminology (112), this corresponds to opposite  $\theta$  angles for  $SU(2)_L$  and  $SU(2)_R$ :  $\theta_L = -\theta_R = \pi$ . Again, discrete symmetries, e.g.  $Z_2 = O(4)/SO(4)$ , which maps  $R \leftrightarrow L$ , and time-reversal, are required to fix these  $\theta$  angles from flowing.

### C. Bulk parton construction for boson SPT with $SO(5) \times Z_2^T$ symmetry: Formal derivation

In this section, we reconsider from a more formal standpoint the parton construction of the 3+1D  $SO(5) \times Z_2^T$  boson SPT presented at the end of section VIC. We will show that this construction precisely recovers the bulk SPT “discrete  $\theta$ -angle” response in Eq. (136). This provides a more physical motivation for our 3 + 1D bulk “completion” of QCD<sub>3</sub> in Eq. (137).

As in section VIC, we begin by considering a 3+1D “topological superconductor” of fermions with  $SO(8) \times Z_2^T$  symmetry. We will then gauge an  $SU(2)$  subgroup and show that the result is precisely the boson SPT of interest. As explained in section VIC, we represent the  $SO(8) \times Z_2^T$  symmetric topological superconductor by eight massive Majorana fermions with an inverted mass,

Eq. (110). Let us consider the partition function on a manifold  $X_4$  in the presence of a background  $SO(8)$  gauge bundle  $A^8$ . We will restrict ourselves to closed oriented manifolds. The partition function will take the form (see appendix E)

$$Z_{TSC}[A^8] = |Z_{TSC}| e^{i\pi \left( \frac{p_1[A^8]}{2} - \frac{\sigma}{2} \right)} \quad (143)$$

with the  $p_1$ ,  $\sigma$  given by Eqs. (124), (125). Further by the Atiyah-Singer index theorem (120), we have

$$\frac{p_1[A^8]}{2} - \frac{\sigma}{2} = \mathcal{J} \in \mathbb{Z}. \quad (144)$$

Here  $2\mathcal{J}$  is the index of the 3 + 1D massless Dirac operator on  $X_4$ . This implies that the partition function in Eq. (143) is real (but not necessarily positive) as required for a time-reversal invariant SPT state.

Note that on a spin manifold  $\frac{p_1[A^8]}{2}$  is itself an integer (and so is  $\frac{\sigma}{2}$ ). This follows because by (127),  $p_1[A^8] = w_2[A^8] \cup w_2[A^8] \pmod{2} = w_2[A^8] \cup w_2[TX_4] = 0 \pmod{2}$ . However, as we eventually will want to describe a boson SPT it is important to be able to formulate the theory on a non-spin manifold. As usual, since the fermions see transition functions in  $(SO(8) \times Spin(4)_{TX_4})/Z_2$ , we only require these transition functions to satisfy the cocycle condition. Then though  $\frac{p_1[A^8]}{2}$  and  $\frac{\sigma}{2}$  are not separately integers, their sum is.

Now, to construct the boson SPT phase, we gauge the  $SU(2)_g$  subgroup of  $SO(8)$ , as in Eq. (111). We further couple the system to a background  $SO(5)$  gauge field  $A^5$ . As we discussed in section VIC, the partons  $\chi$  transform as  $Sp(4)$  spinors under  $SO(5)$ , so they see a combined  $(Sp(4) \times SU(2)_g)/Z_2$  gauge field. Together with the  $Spin(4)_{TX_4}$  transition functions, this yields overall transition functions in  $(Sp(4) \times SU(2) \times Spin(4)_{TX_4})/(Z_2 \times Z_2)$ . This gives the by now familiar condition:

$$w_2[A^5] + w_2[a_g] + w_2[TX_4] = 0 \pmod{2} \quad (145)$$

For such an  $SO(8)$  bundle we have

$$p_1[A^8] = p_1[A^5] + p_1[a_g] \quad (146)$$

Thus, applying Eq. (134), we obtain:

$$Z_{TSC}[A^5, a_g] = |Z_{TSC}| e^{i\pi \int_{X_4} w_4[A^5]} \quad (147)$$

Thus far we treated  $a_g$  as a background gauge field. Now we make it dynamical, *i.e* we integrate the partition function over  $a_g$ . There is no topological term for  $a_g$  and its dynamics will be governed



by the usual Yang-Mills action, which is expected to confine all fields charged under  $a_g$  leaving behind a trivial gapped vacuum. The resulting theory has a partition function with a phase given precisely by  $e^{i\pi \int_{X_4} w_4[A^5]}$ . Further it can be formulated on a non-spin manifold with only a background  $SO(5)$  gauge field. The local operators are bosonic and transform non-projectively under  $SO(5)$ . Thus, we have constructed the desired boson SPT with  $SO(5) \times Z_2^T$  symmetry, and the partition function matches Eq. (136) proposed in section VII B based on consistency arguments.

#### D. Chiral spin-liquid

We note that if we break time-reversal in  $QCD_3$  with an  $Sp(4)$  preserving mass term

$$\delta L = m \bar{\chi} \chi \quad (148)$$

and make the mass  $m$  large enough (compared say to the gauge coupling), we will drive the system into a topologically ordered phase. Integrating the gapped  $\chi$ s in Eq. (137) out:

$$\begin{aligned} S_{QCD_3}^{\text{bulk+bound}} &= \text{sgn}(m) \left[ \frac{i}{2} C S_{SO(5)}[A^5, Y_4] + i C S_{SU(2)}[a^g, Y_4] + 4i C S_g[Y_4] \right] \\ &+ \pi i \int_{X_4 \cup \bar{Y}_4} w_4[A^5] \end{aligned} \quad (149)$$

Without loss of generality, choose  $m > 0$ . By looking at the action for  $a_g$ , we see that we get a  $2 + 1D$   $SU(2)_1$  topological order, which is just a semion state  $\{1, s\}$ . The semion  $s$  is just the Majorana  $\chi$ . The chiral central charge  $c = 2 - 1$ , with  $2 = 4 \times 1/2$  coming from the gravitational Chern-Simons term and  $-1$  from integrating  $a_g$  out. The semion  $\chi$  transforms projectively under the  $SO(5)$  symmetry — as an  $Sp(4)$  spinor. In particular, it carries spin-1/2 under  $SO(3)_s$ . Further, the  $SO(5)$  response is given by a Chern-Simons term with level  $k = 1/2$ ; so the level of Chern-Simons response to  $SO(3)_s$  gauge field is also  $1/2$ . We see that this state has all the properties of a chiral spin liquid [68].

Now, it would be a little surprising if the chiral spin-liquid was in the vicinity of the deconfined quantum critical point. If we consider the  $NCCP^1$  formulation (1), then one operator with the same quantum numbers as  $\bar{\chi}\chi$  is  $\epsilon_{\mu\nu\lambda} \partial_\rho f_{\rho\mu}^b f_{\nu\lambda}^b$  with  $f_{\mu\nu}^b = \partial_\mu b_\nu - \partial_\nu b_\mu$ . Given the number of derivatives, one would naively expect this term to be irrelevant in the  $NCCP^1$  model. If  $NCCP^1$  and  $QCD_3$  indeed share the same fixed point, this would then imply that  $\bar{\chi}\chi$  is irrelevant at the  $QCD_3$  fixed point — an unexpected, but not impossible scenario. It would be interesting to determine the scaling

dimension of this operator numerically at the deconfined critical point. In the lattice magnet, it corresponds to the imaginary part of the plaquette ring exchange:

$$\begin{aligned}\bar{\chi}\chi &\sim \frac{-i}{2}(P_{i+\hat{y},i+\hat{x}+\hat{y}}P_{i+\hat{x}+\hat{y},i+\hat{x}}P_{i+\hat{x},i} - h.c.) \\ &\sim \mathbf{S}_{i+\hat{x}+\hat{y}} \cdot (\mathbf{S}_i \times \mathbf{S}_{i+\hat{x}}) + \mathbf{S}_{i+\hat{y}} \cdot (\mathbf{S}_i \times \mathbf{S}_{i+\hat{x}}) + \mathbf{S}_{i+\hat{y}} \cdot (\mathbf{S}_{i+\hat{x}} \times \mathbf{S}_{i+\hat{x}+\hat{y}}) + \mathbf{S}_{i+\hat{y}} \cdot (\mathbf{S}_i \times \mathbf{S}_{i+\hat{x}+\hat{y}})\end{aligned}$$

where  $P_{ij} = 2\mathbf{S}_i \cdot \mathbf{S}_j + 1/2$  is the exchange operator.

### VIII. DISCUSSION AND IMPLICATIONS OF THE DUALITIES

The most fundamental question about both the  $SU(2)$  invariant and the easy-plane  $NCCP^1$  theories is whether they describe CFTs in the IR. We have not tackled this question head on in this paper. We first discuss what follows if the dualities are assumed to hold in their strong forms in the IR. As mentioned in the introduction, the various theories could fail to flow to nontrivial fixed points. In this scenario the dualities may still be relevant to the ‘quasiuniversal’ physics up to a large lengthscale. We discuss these issues, and what is known about the IR fate of the deconfined critical transitions, in Sec. IX.

#### A. For deconfined criticality

Many consequences of the emergent  $SO(5)$  symmetry have been explored numerically in Ref. [15]. In section IVE we discussed an additional consequence for the phase diagram in the presence of perturbations that break  $SO(5)$ . This will be interesting to explore in future numerical work.

We argued that the proposed duality web provides an explanation of this emergent  $SO(5)$  symmetry, despite the fact that the  $SO(5)$  symmetry is not manifest in any single member of the duality web. In particular the proposed self-duality of the  $SU(2)$  invariant  $NCCP^1$  model immediately implies emergent  $SO(5)$  symmetry in the IR. We will discuss other numerical tests of the fermionic versions of this theory separately below.

We also discussed the continuum  $N_f = 2$   $QCD_3$  theory which has manifest  $SO(5)$  symmetry and which shares the same anomaly as the putative deconfined critical point, and may possibly flow to it in the IR. The IR fate of the  $QCD_3$  theory is not currently known and is a good target for future numerical work. It will be particularly interesting to see if it shares the (quasi)universal power law correlations seen in other models equivalent to the  $SU(2)$ -invariant  $NCCP^1$ .

For the easy plane model direct numerical simulations of quantum magnets find a first order transition. As we have emphasized several times, the nature of the transition in this model is worth revisiting. We have seen that this model is dual to a version of fermionic  $N_f = 2$   $QED_3$  with  $U(1) \times U(1)$  symmetry. For fermionic  $QED_3$  with  $SU(2)$  flavor symmetry, there is some recent evidence that the theory is conformal in the IR.[43] Further the results do not seem to be sensitive to whether the lattice regulator employed actually preserves full  $SU(2)$  flavor symmetry or whether it only has  $U(1)$  flavor symmetry. In light of all this, more numerical studies of the models in the easy plane duality web are clearly called for.

The strongest form of the duality web of these theories asserts that all these theories flow in the IR to the same  $O(4) \times Z_2^T$ -invariant CFT. Below we will describe some implications of the enhanced symmetry expected in such a putative critical theory.

If an  $O(4) \times Z_2^T$  symmetric fixed point does exist then, for the easy plane  $NCCP^1$  to flow to it in the IR it must be that perturbations that break the symmetry to  $(U(1) \times (U(1) \rtimes Z_2)) \times Z_{2T}$  are irrelevant. As discussed in Sec. III C, the simplest such perturbation is a Néel/VBS anisotropy which lies in the  $(2, 2)$  representation of  $SO(4)$  (the quadrupled monopole operator  $\Phi_2^4 + \Phi_2^{*4}$  pertinent to the lattice magnet lies in the same representation). Thus for enlarged  $O(4)$  symmetry we need the scaling dimension  $\Delta_{(2,2)} > 3$  at the  $O(4)$  fixed point.

If an  $O(4) \times Z_2^T$  symmetric fixed point exists, and the strongest form of the duality web holds, then a square lattice spin-1/2 quantum magnet with  $XY$  symmetry can show a direct continuous Neel-VBS transition with enlarged  $O(4)$  symmetry.

An alternate possibility is that the  $O(4) \times Z_2^T$  CFT exists and that  $SU(2)$  flavor symmetric  $QED_3$  flows to it, but the  $U(1) \times U(1)$  theories (easy plane  $NCCP^1$  and  $QED_3$  with the same symmetry) do not flow to that fixed point. This scenario can be tested by numerical simulations of the  $SU(2)$  symmetric  $QED_3$  theory. We detail below how to test for emergent  $O(4)$  symmetry. Should such a fixed point be found, it will be interesting to calculate the scaling dimension of operators transforming under the  $(2, 2)$  representation to test for relevance.

Finally it is possible that an  $O(4) \times Z_2^T$  fixed point of the kind we have described does not exist in the first place. Therefore we next turn to the fermionic theories where this question is best addressed numerically.

Useful analytical insights will also come from conformal bootstrap [24, 25, 82–84]. Note also that the duality webs open up the possibility of analytical results for deconfined critical points using large  $N$  in the fermionic language [85–87].

### B. For QED<sub>3</sub> and QED–GN

The strong self-duality for SU(2) flavor symmetric QED<sub>3</sub> implies an emergent  $O(4)$  symmetry which leads to simple testable predictions.

The fermion bilinears  $\bar{\psi}\sigma\psi$  are expected to be scaling fields, with dimension smaller than their engineering dimension of two, transforming in the (1,1) representation of  $SO(4)$ .  $O(4)$  symmetry relates them to strength-2 monopole operators in the QED<sub>3</sub> theory, so calculating correlations of monopole operators will allow interesting tests of the emergent symmetry. Presumably this requires some modifications of existing numerical calculations of correlators in the QED<sub>3</sub> theory. We therefore also describe several tests using more ordinary correlators.

It should be fruitful to focus on correlations of the conserved  $SO(4)$  currents. The operator  $\bar{\psi}\gamma^0\sigma\psi$  (the time component of one of the SU(2) currents) was already studied in Ref. [43] and shown to have the expected scaling dimension 2. One of the three currents of the other SU(2) is a simple operator in QED<sub>3</sub>: this is the gauge flux  $\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda$ . Therefore its time component, the magnetic flux, is related by symmetry to  $\bar{\psi}\gamma^0\sigma\psi$ . Right at the critical point these operators should have scaling dimension 2; this follows from their conservation and is not a test of the symmetry rotating them. A simple consequence of emergent  $O(4)$  is that the universal amplitudes of the two point functions should also be the same for these different currents. Specifically if we compare the correlators of the SU(2) currents with the correlators of  $\frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda$ , they should have the same universal amplitude in addition to the same scaling dimension.

A more dramatic consequence arises if we perturb the critical point by turning on either a non-zero temperature  $T$  or a fermion mass  $m$ , both of which preserve  $SO(4)$ . Then the current correlations will involve a non-trivial universal scaling function

$$|k|F\left(\frac{m}{T}, \frac{\omega}{|k|}, \frac{|k|}{m}\right). \quad (150)$$

Now  $SO(4)$  symmetry predicts that this scaling function is identical for the SU(2) current and for the 3-flux of the gauge field. It will be very interesting to test this. For instance the SU(2) spin susceptibility should be described by the same crossover function as the diamagnetic susceptibility of the gauge field, and likewise the SU(2) phase stiffness should be described by the same crossover function as the Meissner stiffness of the gauge field.

Finally, a representative of the important (2,2) operator will be given by, *e.g.*,  $2(\bar{\psi}\sigma^z\psi)^2 - (\bar{\psi}\sigma^x\psi)^2 - (\bar{\psi}\sigma^y\psi)^2$  (we are assuming that the other  $SO(4)$  representations contributing

to this operator are less relevant, as expected from the discussion in Sec. III C). If  $O(4)$  symmetry is established numerically, then the irrelevance of this  $O(4)$ -breaking perturbation can be tested.

For the  $\text{QED}_3$ -GN model, the first issue that should be addressed numerically is whether the transition is second order at all (the duality with the  $NCCP^1$  model suggests there should be critical behaviour at least up to a large length scale). Should such a second order transition be found, a number of its properties can be predicted using our results.

First, if we measure  $\phi$  correlations at this fixed point, we are measuring correlations of the  $\text{SO}(5)$  vector. They can therefore be compared with the Néel and VBS correlation functions known from  $NCCP^1$  simulations. Second, the  $\phi^2$  operator takes us to the  $\text{QED}_3$ /easy plane  $NCCP^1$  fixed point. We know that with  $\text{SO}(5)$  this is in the same representation as the operator that tunes through the transition in the  $NCCP^1$  theory (a component of  $X^{(2)}$  in the notation of Sec. IV D). Hence the  $\phi^2$  scaling dimension can be compared with results for  $\nu$  at the  $\text{SU}(2)$ -symmetric deconfined critical point.

More interestingly, the fermion bilinear  $\bar{\psi}\sigma^z\psi$  also corresponds to an element of  $X^{(2)}$ . Thus the vector  $\bar{\psi}\sigma\psi$  should have the same correlations as  $\phi^2$  at the  $\text{QED}_3$ -GN fixed point (modulo subleading contributions) if there is full  $\text{SO}(5)$  symmetry. This last statement is particularly interesting as it does not involve comparing with a different theory — both quantities are calculated in the same simulation.

### C. Comparison between the $N = 2$ $\text{QED}_3$ , bilayer honeycomb lattice model, and easy plane spin models

For the putative  $O(4)$  fixed point, there are (at least) three lattice model realizations that can be (and have been) studied numerically:  $N = 2$  lattice  $\text{QED}_3$ , spin models that realize the easy-plane deconfined transition (if a model with a second order transition exists), and the bilayer honeycomb lattice interacting fermion model, studied in Ref. [44, 45], that realizes the transition between a trivial and SPT boson insulator with explicit  $\text{SO}(4)$  symmetry. The critical exponents measured in different models should be related to each other, which we discuss below.

The  $N = 2$   $\text{QED}_3$  was treated as a stable CFT in Ref. [43], so there is no correlation length critical exponent. But there is still the anomalous dimensions associated with the mass operators

$$M_z = \bar{\psi}_1 \psi_1 - \bar{\psi}_2 \psi_2, \quad M_0 = \bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2,$$

$$\langle M_z(0) M_z(r) \rangle \sim \frac{1}{r^{1+\eta_{\bar{\psi}\sigma^z\psi}}}, \quad \langle M_0(0) M_0(r) \rangle \sim \frac{1}{r^{1+\eta_{\bar{\psi}\psi}}} \quad (151)$$

According to Ref. [43],  $\eta_{\bar{\psi}\sigma^z\psi} \sim 1.0$ . To our knowledge, a careful study of  $\eta_{\bar{\psi}\psi}$  has not been performed in numerical simulations of QED<sub>3</sub> - we hope that future simulations will also address this exponent.

The bilayer honeycomb lattice model describes a bosonic transition, which may potentially also be described by the  $N = 2$  QED<sub>3</sub>.<sup>42</sup> The tuning parameter for this transition corresponds to the fermion mass  $m(\bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2)$  in the field theory. There is a correlation length exponent  $\nu_{bh}$  defined as

$$\xi \sim m^{-\nu_{bh}} \sim (J - J_c)^{-\nu_{bh}}, \quad (152)$$

where  $J$  is the interaction on the lattice that is tuned to the critical point. The O(4) order parameter  $n_a$  has an anomalous dimension  $\eta_{bh}$ :

$$\langle n_a(0) n_a(r) \rangle \sim \frac{1}{r^{1+\eta_{bh}}}. \quad (153)$$

The easy-plane spin models have three different exponents:  $\eta_{xy}$  (the same as  $\eta_{vbs}$ ),  $\eta_z$  and  $\nu_{jq}$ :

$$\xi \sim (Q - Q_c)^{-\nu_{jq}}, \quad \langle S^x(0) S^x(r) \rangle \sim \frac{(-1)^r}{r^{1+\eta_{xy}}}, \quad \langle S^z(0) S^z(r) \rangle \sim \frac{(-1)^r}{r^{1+\eta_z}}. \quad (154)$$

Where  $Q$  is a tuning parameter for the transition. If the strong duality holds, we have the following relations:

$$3 - \frac{1}{\nu_{jq}} = \frac{1 + \eta_{\bar{\psi}\sigma^z\psi}}{2}, \quad \eta_z = \eta_{\bar{\psi}\psi}, \quad \eta_{xy} = \eta_{bh}, \quad 3 - \frac{1}{\nu_{bh}} = \frac{1 + \eta_z}{2}. \quad (155)$$

## IX. CRITICAL AND PSEUDOCRITICAL POINTS

It is not yet certain whether the SU(2)-symmetric  $NCCP^1$  model has a true critical point, or whether it instead shows ‘pseudocritical’ behaviour with a very large but finite correlation length. Here we review what is currently known from simulations, and clarify what the latter possibility would mean for the dualities presented here. We also briefly discuss the easy plane-case.

<sup>42</sup> Since the microscopic model has an exact  $SO(4)$  symmetry here, for QED<sub>3</sub> to describe it, the  $SO(4)$  symmetry must necessarily emerge at the critical point.

Various lattice models that show a phase transition ‘in the  $NCCP^1$  universality class’ have been studied numerically [6–22].<sup>43</sup> The basic feature of these simulations is that the correlation length  $\xi$  appears to diverge as the critical point is approached, certainly becoming larger than numerically accessible system sizes (up to 640 lattice spacings in the model of Ref. [14]). At these lengthscales the standard signs of first-order behaviour, e.g. double-peaked probability distributions, are absent. The qualitative features of the transition are as expected from the theory of deconfined criticality,<sup>44</sup> and finite size estimates of critical exponents are roughly consistent between different lattice models.

These features are consistent with a continuous transition (which much recent work assumes). However, it was noted some time ago that various naively ‘universal’ quantities instead drift with system size, leading to controversy about whether the transition was ultimately continuous or first order [10–13, 17]. Ref. [14] argued that these drifts are not merely conventional finite-size corrections to CFT scaling behaviour, since making this assumption leads to unphysical negative values for the anomalous dimensions at large sizes, and suggested two possible scenarios for reconciling the various numerical results. One scenario is that the  $NCCP^1$  model shows a continuous transition, but with unconventional finite-size scaling behaviour due to a dangerously irrelevant variable<sup>45</sup> (see also [23, 89]). The second scenario is that  $NCCP^1$  shows a first order transition [12, 13, 17] which is rendered anomalously weak by a quasi-universal mechanism [14] which we discuss below.

Further complicating the issue, it was found numerically that critical fluctuations at the deconfined transition are  $SO(5)$  symmetric to a high level of precision [15]. Level degeneracies found in the JQ model [91] also support this enhanced symmetry (the approximate equality of Néel and VBS scaling dimensions had been noticed earlier by Sandvik [92]). At first sight  $SO(5)$  symmetry seems to be strong evidence that the critical  $NCCP^1$  model flows to an  $SO(5)$ –invariant CFT. However, subsequent investigations [24, 25] of  $SO(5)$ –symmetric CFTs using the conformal bootstrap [82, 83] did *not* find a sufficiently stable<sup>46</sup> CFT in the expected region of parameter space. The bootstrap

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<sup>43</sup> These include the J–Q model [6–13] and a related loop model [14, 15], lattice [16, 17] and continuum [18] field theories, and the classical 3D dimer model [19–22], which has different microscopic symmetries to the JQ model but has been argued to have the same continuum description. See also numerical work on generalizations to  $SU(n)$  with  $n > 2$  [88–90].

<sup>44</sup> For example, in models for the Néel–VBS transition there is a single direct transition between the two phases, the expected  $U(1)$  symmetry for the VBS order parameter indeed emerges at the critical point, and the Néel vector has a large anomalous dimension, which was predicted as a signature of deconfinement.

<sup>45</sup> Some features of the numerical results are suggestive of this. See e.g. the discussion of correlation functions in [14], and the fits in [23].

<sup>46</sup> To describe the (generic) deconfined critical point, the  $SO(5)$  CFT should have no relevant operator that is invariant

shows that any sufficiently stable  $\text{SO}(5)$ -invariant CFT must have a larger anomalous dimension for the  $\text{SO}(5)$  vector than is expected from simulations of deconfined criticality [24, 25]. In view of this, it makes sense to revisit the weakly-first-order scenario with  $\text{SO}(5)$  symmetry in mind.

At first sight a first order transition with  $\xi \gg 1$  is implausible because of a fine-tuning problem. If a theory has no nontrivial stable fixed point, the obvious way to get a large  $\xi$  is to fine-tune it close to an unstable fixed point.<sup>47</sup> Since this mechanism relies on fine-tuning, it is unlikely to be the explanation for the apparent critical behaviour at the DCP, which seems to be generic. However there is an alternative generic mechanism for ‘pseudocritical’ behaviour with very large  $\xi$  [93–95]. In this scenario, the large  $\xi$  can be understood in terms of a fixed point which exists slightly outside the physical parameter space of the model — for example at slightly smaller spatial dimension  $d_c$ . The structure of the RG flows close to  $d_c$  implies an exponentially large correlation length for  $d \gtrsim d_c$ . This mechanism depends on an accident in the universal structure of the RG flows, but it does not require fine-tuning of a given microscopic Hamiltonian. Additionally, this scenario is plausible for the  $NCCP^1$  model (and indeed  $NCCP^{n-1}$  for nearby values of  $n$ ), given what is known about the  $d$ -dimensional  $NCCP^{n-1}$  model in various limits [14].

The basic mechanism is the annihilation of a stable and an unstable fixed point as a parameter  $\tau$  is varied. Here  $\tau$  is a quantity which does not flow under RG, such as the spatial dimension (in the case of  $NCCP^1$ ) or the rank of a symmetry group. Quite generally, close to  $\tau_c$  the RG equation for the coupling  $\lambda$  which is becoming marginal looks like

$$\frac{d\lambda}{d \ln L} = a(\tau_c - \tau) - \lambda^2, \quad (156)$$

where  $a$  and  $\tau_c$  are universal constants and  $a > 0$ . For  $\tau < \tau_c$  both fixed points exist and for  $\tau > \tau_c$  neither do. But for  $\tau \gtrsim \tau_c$  the RG flows become very slow close to  $\lambda = 0$ : the long RG time required to traverse the ‘pseudocritical’ region corresponds to a large lengthscale  $\xi \sim \ell_0 \exp(\pi/\sqrt{a(\tau - \tau_c)})$ , where  $\ell_0$  is nonuniversal. The large amount of RG time spent near  $\lambda = 0$  implies that the properties of the pseudocritical regime are quasiuniversal in the limit of small  $\tau - \tau_c$ .

In more detail, this is because (in the formal limit of small  $\tau - \tau_c$ ) the *subleading* RG couplings,

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under all symmetries, see Sec. IV D. The bootstrap result constrains the  $\text{SO}(5)$  vector’s anomalous dimension, under the assumption that there is no relevant symmetry-trivial operator.

<sup>47</sup> Take for example the  $q$ -state Potts model in 3D, with a large value of  $q$ . The transition is generically first order due to a cubic invariant in the Landau-Ginsburg action. One could fine-tune the couplings to be close to the free fixed point, giving a very weak first order transition. This mechanism is non-generic, and is not the mechanism discussed in the text.



$g_i$ , have time to flow to well-defined pseudocritical values, independent of their bare values in a given microscopic model. (The relevant coupling which drives the transition is zero since we consider the theory in the critical plane.) The RG flow is attracted to a quasiuniversal trajectory through coupling constant space, given by setting  $g_i = 0$  up to corrections that are exponentially small in  $1/\sqrt{\tau - \tau_c}$ . Note that ‘quasiuniversality’ holds to exponentially good precision in  $1/\sqrt{\tau - \tau_c}$ , despite the fact that the flow of  $\lambda$  during a stretch of RG time of order  $\ln \xi$  is larger than this.<sup>48</sup> (This will lead to quasiuniversal drifts in e.g. effective exponents.)

The  $Q$ -state Potts model in 2D provides an example of this phenomenon with  $\tau = Q$  [93–95, 97–100] (in this context  $\lambda$  was originally thought of as a fugacity for Potts vacancies [93]). For  $Q < 4$  both a critical and a tricritical point exist, and they merge at  $Q = 4$ . For  $Q \gtrsim 4$ , the Potts transition is very weakly first order. A priori the above picture applies only for  $(Q - 4) \ll 1$ , but empirically it is found that the transition remains weakly first order at least for  $Q = 5, 6, 7$ , where  $\xi \simeq 2512, 159, 48$  respectively on the square lattice [100]. This mechanism for generating a small mass scale has also been discussed in the context of 4D QCD [101, 102], with  $\tau = -N_f/N_c$ .<sup>49</sup> In the  $NCCP^{n-1}$  model, it is plausible that there is a range of  $n$  where the transition is weakly first order but can be rendered continuous by slightly decreasing the spatial dimension.

It should be noted that the choice of deformation parameter  $\tau$  is not unique; for example in the weakly first order regime of the Potts model the transition can be made continuous by reducing either  $Q$  or  $d$  (and in  $NCCP^{n-1}$  we can certainly render the transition continuous by a large enough increase in  $n$ ). Alternately, one may consider the theory only at the physical value of  $\tau$ , and attribute the pseudocritical behaviour to proximity to the nonunitary fixed points at  $\lambda = \pm i\sqrt{a(\tau - \tau_c)}$ .

A possible explanation for the various numerical results for the deconfined transition is that there

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<sup>48</sup> We may see this in more detail using the logic of [95, 96]. Starting with the RG equations for  $\lambda$  and for the leading irrelevant coupling  $g$  at  $\tau = \tau_c$ , expand in  $\Delta^2 = \tau - \tau_c$  assuming analyticity in  $\Delta^2$  and the couplings. Using the freedom to make analytic redefinitions of the couplings gives (156) and  $dg/dt = -(y + m\lambda)g$ , neglecting subleading corrections ( $t$  is RG time). The zero of the latter equation at  $g = 0$  is preserved to all orders in  $\Delta^2$ . We have  $\lambda(t) = -\Delta \tan \Delta[t - t_*]$  with  $t_* \simeq \pi/2\Delta - 1/\lambda(0)$ . The correlation length is determined by setting  $\lambda \sim -1$ , giving  $\ln \xi = \pi/\Delta + O(1)$ . The subleading coupling behaves as  $g(t) = g_0 e^{-yt} \left( \frac{\Delta^2 + \lambda(t)^2}{\Delta^2 + \lambda(0)^2} \right)^{m/2}$ . Once the RG time is of order  $1/\Delta$ ,  $g(t)$  has become exponentially small in  $1/\Delta$ . On RG timescales of this order the typical variation in  $\lambda(t)$ , and therefore the typical ‘quasiuniversal’ drift in large scale properties, is  $O(\Delta)$ .

<sup>49</sup> A fixed point annihilation phenomenon also occurs in the so called ‘compact’  $CP^{n-1}$  model (an  $SU(n)$ -symmetric Landau-Ginsburg theory obtained from  $NCCP^{n-1}$  by condensing the strength-1 monopole). However there the critical and tricritical fixed points annihilate when  $n$  is *increased* (with  $n_c \sim 3$ ) rather than when  $n$  is *decreased* as is the case for  $NCCP^{n-1}$  [103].

is a pseudocritical regime within the  $\text{SO}(5)$ -symmetric subspace of theory space. If the effective scaling dimensions of allowed  $\text{SO}(5)$  breaking perturbations — specifically, the symmetric tensor  $X_{abcd}^{(4)}$  discussed in Sec. IV D — are greater than three, the  $NCCP^1$  model and QED–Gross-Neveu can lie in the basin of attraction of this regime, and will also show pseudocritical behaviour. In this scenario the dualities we have discussed apply to the physics at lengthscales up to  $\xi$  (and somewhat beyond, see below).

To make the above possibility more concrete we may think of  $NCCP^1$  as a perturbation of an exactly  $\text{SO}(5)$ -invariant theory whose RG behaviour could in principle be pinned down. The nonlinear sigma model is one possibility: in the above scenario we would expect the sigma model at strong coupling to exhibit pseudocritical behaviour. However the language of the sigma model does not give us an obvious candidate for the deformation parameter  $\tau$ . Tentatively, a renormalizable alternative may be the  $N_c = 2$ ,  $N_f = 2$  QCD<sub>3</sub> discussed in Sec. VI. This theory has an  $\text{SO}(5)$  symmetry which becomes explicit when it is written in terms of Majorana fermions. The theory may spontaneously break  $\text{SO}(5)$  on the scale set by the coupling, in which case it is not very interesting. But another possibility is that it generates a long lengthscale by the above mechanism. If so,  $N_f$  and  $d$  are candidates for deformation parameters  $\tau$  which could produce a true fixed point.

Another way to think about the possibility of an  $\text{SO}(5)$ -invariant pseudocritical regime is to hypothesize an exact  $\text{SO}(5)$  symmetry for the nearby nonunitary fixed points at imaginary  $\lambda$ ; we thank S. Pufu for suggesting this. We would then view the  $NCCP^1$  model and QED–Gross-Neveu on lengthscales  $\lesssim \xi$  as perturbations away from this fixed point.

As an aside, note that in the present scenario  $\text{SO}(5)$  symmetry survives to lengthscales even larger than  $\xi$ . The simplest possibility is that on scales larger than  $\xi$  the system flows to the ordered phase of the  $\text{SO}(5)$  sigma model, representing a first order transition for  $NCCP^1$  — although in principle it is possible that the theory could flow to a new nontrivial fixed point. This ordered sigma model is subject to anisotropies due to the  $\text{SO}(5)$ -breaking perturbations that are allowed in the microscopic model. However these anisotropies are small since the effective RG eigenvalue  $y_4$  of the 4-index symmetric tensor is negative in the pseudocritical regime: they do not become apparent until a lengthscale  $L_* \sim \xi^{1+|y_4|/3} \times (\text{bare coupling})$ . Simulations in the range  $\xi \lesssim L \lesssim L_*$  would find four apparent Goldstone modes.<sup>50</sup>

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<sup>50</sup> It is clear from the drift in effective critical exponents that current simulations are not in this regime (which would show simple exponent values).

A similar conjecture could in principle apply to the dualities between  $O(4)$  invariant theories, although numerical results for the easy-plane Néel-VBS transition suggest that a first order transition with a rather shorter correlation length than in the  $SU(2)$  case may be generic there [20, 57–60]. As noted in Sec. III C, it is also conceivable that the fermionic theories flow to an  $O(4)$  symmetric fixed point (or pseudocritical point), but that the easy plane model does not, as a result of an additional perturbation allowed by microscopic symmetry. Further numerical studies of both the easy plane model and of  $QED_3$  along the lines described in Sec. VIII are clearly called for.

For quantum phase transitions, the pseudocriticality scenario implies that the system will show quantum critical behavior above a parametrically low temperature scale  $T^* \sim J \exp\left(-\pi/\sqrt{a(\tau - \tau_c)}\right)$ , where  $J$  is a microscopic energy scale, with critical exponents drifting as the temperature changes. Criticality eventually disappears below  $T^*$ , and the system possibly crosses over to a first order transition. But for sufficiently low  $T^*$ , a quantum critical regime (the famous ‘critical fan’) should be observable above  $T^*$ . Pseudocritical systems thus present interesting possibilities for phenomenology near quantum phase transitions.

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## Appendix A: More precise Lagrangians

In this Appendix we present the easy-plane dualities in a more precise notation. We define a Dirac fermion through a Pauli-Villars regulator, such that its partition function (under a general  $U(1)$  gauge field  $a$  and metric  $g$ ) is given by  $Z_\psi = |Z_\psi|e^{-i\pi\eta[a,g]/2}$ . In the usual notation (used in the main text), this would corresponds to a Dirac Lagrangian supplemented with a Chern-Simons term at level  $k = -1/2$  for both  $a$  and  $g$ . In general it is more precise to use the  $\eta$ -invariant instead of the less well-defined  $k = -1/2$  Chern-Simons terms (for more details, see, for example, Ref. [54] for a review).

We also enforce the proper quantization of all the  $U(1)$  gauge field (including dynamical and background ones) from the beginning, by working directly with  $B_{1,2}$  (instead of  $B, B'$ ) in the main text. The duality now reads:

$$\begin{aligned} & |D_b z_1|^2 + |D_{b-B_1} z_2|^2 - |z_1|^4 - |z_2|^4 + \frac{1}{2\pi} b d B_2 \\ \iff & \bar{\psi}_1 i \not{D}_a \psi_1 + \bar{\psi}_2 i \not{D}_{a+B_2-B_1} \psi_2 + \frac{1}{4\pi} a d a + \frac{1}{2\pi} a d B_2 + \frac{1}{4\pi} B_2 d B_2 + 2CS_g, \end{aligned} \quad (A1)$$

where  $CS_g$  is a gravitational Chern-Simons term, normalized such that in the absence of any other matter field, it leads to a thermal hall conductance  $\kappa_{xy} = \frac{1}{2} \frac{\pi^2 k_B^2 T}{3h}$ .

## Appendix B: Some other dualities

Here we describe an alternate set of dualities between theories with global  $SU(2)$ ,  $U(1)$  and  $\mathcal{T}$  symmetries. We begin with a duality between  $N = 2$  species of a free massless Dirac fermion<sup>51</sup> and a bosonic theory.

To be concrete, we define the partition function of the massless Dirac fermion in terms of the  $\eta$ -invariant. To maintain  $SU(2)$  symmetry between the two species we must choose the same regularization for both species. Therefore we write the partition function of the  $N = 2$  free massless Dirac fermion as

$$Z_\psi = |Z_\psi|e^{-i\pi\eta[A,g]} \quad (B1)$$

Here  $A, g$  are the background gauge field (strictly speaking a  $\text{spin}_c$  connection) and metric respectively. We know that  $e^{-i\pi\eta[A,g]} = e^{-i \int \frac{1}{4\pi} A d A + 2CS[g]}$ . Therefore the theory can be made time reversal

<sup>51</sup> In this and the next Appendix we use the precise definition of the Dirac partition function in terms of the  $\eta$ -invariant. We will also include a coupling to a background space-time metric  $g$ .

invariant (while keeping  $SU(2)$  and  $U(1)$ ) by adding  $\int \frac{1}{4\pi} AdA + 2CS[g]$ . Thus we consider<sup>52</sup>

$$\mathcal{L}_{0f} = i\bar{\psi}_\alpha D_A \psi_\alpha + \frac{1}{4\pi} AdA + 2CS[g] \quad (B2)$$

We claim this has a dual bosonic description:

$$\mathcal{L}_{0b} = \mathcal{L}[Z_\alpha, b] + \frac{1}{2\pi} bdA - \frac{1}{4\pi} bdb, \quad (B3)$$

where  $b$  is an ordinary  $U(1)$  gauge field, and  $Z_\alpha$  is a spin-1/2 (under the global internal  $SU(2)$ ) boson. As a check on this proposal, consider giving the fermions a mass that preserves  $SU(2) \times U(1)$ . If  $m < 0$  (with our definition of the fermion determinant) we get  $\int \frac{1}{4\pi} AdA + 2CS[g]$  (which corresponds to a gapped phase with  $\sigma_{xy} = 1, \kappa_{xy} = 1$ ). If  $m > 0$ , then we get the same but with opposite sign ( $\sigma_{xy} = -1, \kappa_{xy}/\kappa_0 = -1$ ). The massless Dirac fermion sits right at the transition between these two phases.

To match these from the boson side, if  $Z_\alpha$  is in a trivial insulator, we integrate it out first, and then integrate out  $b$ . This gives  $\int \frac{1}{4\pi} AdA + 2CS[g]$  which exactly matches the fermion side with  $m < 0$ . The other phase is obtained by putting  $Z_\alpha$  in a boson integer quantum Hall state (bIQH). Then integrating out  $z_\alpha$  gives a term  $\frac{2}{4\pi} bdb$ . Combining with the other terms already in the boson action, and integrating out  $b$ , we get  $-(\int \frac{1}{4\pi} AdA + 2CS[g])$  which also exactly matches the answer from the fermion side with  $m > 0$ .

Therefore what in the boson theory is the transition between a trivial insulator and the bIQH state is dual to the free massless Dirac fermion. This should not be surprising to us. In fact, we already know two different forms of actions that describe bIQH transitions:  $QED_3$  with  $N_f = 2$  and the easy-plane  $NCCP^1$ . If we use the  $QED_3$  Lagrangian in Eq. (B3), we simply get back the free Dirac fermion theory. If we use the easy-plane  $NCCP^1$  theory instead, we obtain a “composite boson” dual of two free Dirac fermions discussed in Ref. [104]. However, in this representation the spin  $SU(2)$  symmetry is non-manifest. Thus in the absence of a manifestly  $SU(2)$ -invariant bosonic representation of the bIQH transition, we will just keep the form of Lagrangian in Eq. (B3) implicit.

We can also check monopole operators. A  $2\pi$  monopole  $\mathcal{M}_b$  of  $b$  carries  $U_A(1)$  charge 1, and  $U_b(1)$  charge  $-1$ . Thus  $Z_\alpha \mathcal{M}_b$  has no  $U_b(1)$  charge, has charge-1 under  $U_A(1)$ , is an  $SU(2)$  doublet and is a fermion. We should identify it with  $\psi_\alpha$ .

Now we follow the usual logic to generate other dualities. First we let  $A \rightarrow a$  and treat  $a$  as a dynamical gauge field (spin<sub>c</sub> connection), and couple  $\frac{1}{2\pi} adB + \frac{1}{4\pi} BdB$  where  $B$  is an ordinary  $U(1)$

<sup>52</sup> We have not shown explicitly background  $SU(2)$  gauge fields - they can be incorporated if needed.

gauge field. The fermion side becomes

$$\mathcal{L}_{1f} = i\bar{\psi}_\alpha \not{D}_a \psi_\alpha + \frac{1}{2\pi} adB + \frac{1}{4\pi} BdB + 2CS[g] \quad (\text{B4})$$

In the boson side the  $a$  dependence arises solely through the term  $\frac{1}{2\pi}(b+B)da$ . Integrating out  $a$  we set  $b = -B$  to get

$$\mathcal{L}_{1b} = \mathcal{L}[Z_\alpha, -B] \quad (\text{B5})$$

The two theories Eq. (B4) and B5 are dual to each other in the following sense. The phase transition in Eqn B5 between the trivial gapped phase and the bIQH state is described by Eq. (B4) when the Dirac fermions are massless. Indeed we used this identification in various parts of the paper.

Now we make  $B$  dynamical  $B \rightarrow b$ , and couple  $\frac{1}{2\pi}bdC$  where  $C$  is an ordinary  $U(1)$  gauge field. In the fermion side the  $b$  dependence occurs through the term  $\frac{1}{2\pi}(a+C)db + \frac{1}{4\pi}bdb$ . Integrating out  $b$  leads to

$$\mathcal{L}_f = i\bar{\psi}_\alpha D_a \psi_\alpha - \frac{1}{4\pi}ada + \frac{1}{2\pi}adC - \frac{1}{4\pi}CdC \quad (\text{B6})$$

The boson side becomes

$$\mathcal{L}_b = \mathcal{L}[Z_\alpha, -b] + \frac{1}{2\pi}Cdb \quad (\text{B7})$$

This almost looks like the  $NCCP^1$  model but we should remember the precise sense in which Eqns. B6 and B7 are dual. As in all the previous examples, the trivial to bIQH transition of  $z_\alpha$  is dual to the massless Dirac theory. Let us interpret this phase transition in the boson side more clearly. When  $z_\alpha$  is in a trivial gapped phase,  $C$  is Higgsed, and we have broken  $U_C(1)$  symmetry (a “superfluid”). When  $z_\alpha$  is in a bIQH state, we have a  $U(1)_2$  theory, and this is really an  $SU(2)$  symmetric chiral spin liquid where the semion is a spin-1/2 spinon.

Thus the transition between a superfluid (that breaks  $U(1)$  but preserves  $SU(2)$ ) and this chiral spin liquid with  $SU(2)$  symmetry is described by Eq. (B6). It is easy to check that this is reproduced by thinking directly about the fermions.

The duality between the theories in Eqns. B7 (interpreted as above) and B6 should be contrasted with the duality of the  $SU(2)$  invariant  $NCCP^1$  model to the  $QED_3$ -GN model. The two sets of dualities describe two distinct phase transitions of the same underlying spin system. Further though both Eqn. B7 has the same  $SO(3) \times U(1)$  symmetry as  $NCCP^1$ , it - unlike  $NCCP^1$  - is not time reversal invariant.

### Appendix C: A different view of the dualities and emergent symmetries

Here we discuss the dualities and emergent symmetries from a point of view familiar in the high energy literature. Caution is however needed as we describe below.

For any  $2 + 1$ D CFT with a global  $U(1)$  symmetry, there is a formal operation on the path integral, denoted  $S$ , which is defined as follows:

$$Z_S[B] = \int \mathcal{D}A \, Z_{\text{CFT}_1}[A] e^{\frac{i}{2\pi} \int d^3x A dB}. \quad (\text{C1})$$

Here  $Z_{\text{CFT}_1}[A]$  is the partition function of the  $2 + 1$ D CFT in the presence of a *background*  $U(1)$  gauge field  $A$ . The operation  $S$  converts this background gauge field into a dynamical one, without including a kinetic term for the field  $A$ . A new background  $U(1)$  gauge field  $B$ , coupling to  $dA/2\pi$  (which is conserved) is also introduced. This operation was defined and used by Kapustin and Strassler [47], and by Witten [48]. A different operation,  $T$ , was also introduced by Witten: this simply shifts the level of the Chern-Simons term for the background gauge field by 1.

If the path-integral on the right hand side of Eqn. C1 is well-defined, then  $Z_S[B]$  is the partition function of a new theory with a new global  $U(1)$  symmetry ( $B$  couples to the current of this symmetry). Further, the theory  $Z_S[B]$  is conformally invariant, at least at the formal level — it is to ensure this that no kinetic term for  $A$  is introduced in the definition of  $S$  — and defines a new conformal field theory which we denote  $\text{CFT}_2$ .

Schematically we write the  $S$  operation as  $S[\text{CFT}_1] = \text{CFT}_2$  where both CFTs have a global  $U(1)$  symmetry. The combination of  $S$  and  $T$  then leads to a remarkable  $SL(2, Z)$  action on the set of  $2 + 1$ D CFTs with a global  $U(1)$  symmetry [48]. (See Refs [105–108] for other appearances of this mathematical structure in related contexts.) That is,  $S$  and  $T$  can be shown formally to satisfy the defining relations<sup>53</sup>  $S^2 = -1$  and  $(ST)^3 = 1$  of  $SL(2, Z)$ .

Let us think a bit more about  $S$ . A priori it is not evident that the path integral in Eq. C1 is well-defined. To obtain some intuition, consider a modified operation which is certainly well-defined:

$$\tilde{Z}_S[B; e^2] = \int \mathcal{D}A \, Z_{\text{CFT}_1}[A] \exp \left( - \int d^3x \left( \frac{1}{2e^2} (dA)^2 - \frac{i}{2\pi} A dB \right) \right). \quad (\text{C2})$$

We have introduced a Maxwell term for the gauge field  $A$  with coupling constant  $e^2$ . Formally, the original  $S$  operation may be written

$$Z_S[B] = \lim_{e^2 \rightarrow \infty} \tilde{Z}_S[B; e^2]. \quad (\text{C3})$$

<sup>53</sup> Here the operation  $S^2 = -1$  corresponds to changing the sign of the gauge coupling.

However Eq. C3 has a more intuitive interpretation. Consider the theory at a fixed value of  $e^2$ . The gauge coupling introduces a length scale  $l_e \sim 1/e^2$ . At distances much smaller than  $l_e$  the physics is that of  $CFT_1$  plus a decoupled free photon, so this is a ‘weakly’ gauged version of  $CFT_1$ . But the coupling between  $CFT_1$  and the photon is relevant, so the physics on distances much larger than  $l_e$  will be different. The limit  $e^2 \rightarrow \infty$  is equivalent to describing the deep IR limit of the  $\tilde{Z}_S$  theory, i.e. distances much greater than  $l_e$ .

At the formal level this deep IR limit is a new conformally invariant theory,  $CFT_2$ , described by  $Z_S$ . Formally, the relationship  $S^2 = -1$  (see below) also requires  $CFT_2$  to be nontrivial if  $CFT_1$  is. But it is not obvious that the conclusions of these formal arguments will always hold in reality, at least for the non-supersymmetric, finite ‘ $N$ ’ theories of interest in this paper. For instance if we take  $CFT_1$  to be the theory of  $N_f$  massless 2-component Dirac fermions ( $N_f$  even) then we obtain  $QED_3$  for  $\tilde{Z}_S$ . Whether or not this flows to a  $CFT$  for general  $N_f$ , not necessarily large, is a long-standing issue which has not yet been settled. For another example relevant to this paper, take  $CFT_1$  to describe a pair of boson fields, each separately at the  $U(1)$  Wilson Fisher fixed point, and take  $S$  to act on the diagonal  $U(1)$  symmetry.  $\tilde{Z}_S$  then describes the easy plane  $NCCP^1$  model in a particular limit.<sup>54</sup> As discussed in the main text, it is hardly clear that this flows to a CFT in the IR. Similarly, if we start with the  $O(4)$  Wilson-Fisher theory and use  $S$  to gauge an appropriate  $U(1)$  subgroup, we obtain the  $SU(2)$ -symmetric  $NCCP^1$  model (in a similar limit). Whether or not this flows to a CFT is again a nontrivial question.<sup>55</sup> (Even when there is a flow to a CFT, we might expect to have to tune the coupling of any relevant symmetry-allowed operators of  $CFT_1$  in order to be on this flow line, contrary to the expectation from the formal limit.)

What about the crucial relationship  $S^2 = -1$ ? We apply the limiting procedure twice to give the partition function

$$Z_{S^2}[C] = \lim_{e'^2 \rightarrow \infty} \lim_{e^2 \rightarrow \infty} \tilde{Z}_{S^2}[C; e^2, e'^2], \quad (C4)$$

<sup>54</sup> This limit is where the gauge coupling is much weaker than the quartic terms (so that the RG flow comes very close to the ungauged Wilson-Fisher fixed point before the gauge coupling drives it away). Also, in this example the bare quartic terms do not couple the two fields.

<sup>55</sup> Note though that if we *assume* the reliability of the  $S$  operation then we can infer the existence of a CFT with the same symmetries, etc. as  $NCCP^1$ . It should be noted however that the formal construction says nothing about the stability (number of relevant perturbations) of this CFT.



with

$$\tilde{Z}_{S^2}[C; e^2, e'^2] = \int \mathcal{D}B \int \mathcal{D}A Z_{\text{CFT}_1}[A] \exp \left( - \int d^3x \left( \frac{(dA)^2}{2e^2} + \frac{(dB)^2}{2e'^2} - \frac{i}{2\pi} B d(C + A) \right) \right). \quad (\text{C5})$$

The proof in Ref. [48] of  $S^2 = -1$  evaluates the path integral above in the absence of the Maxwell terms (as appropriate to the formal definition of  $S$  in Eqn. C1). The  $B$  integral then acts as a delta function enforcing  $C = -A$ , and the right hand side becomes  $Z_{\text{CFT}_1}[-C]$ .

Even in the case where  $l_e \gg l_{e'}$ ,

$$\lim_{e^2 \rightarrow \infty} \lim_{e'^2 \rightarrow \infty} \tilde{Z}_{S^2}[C; e^2, e'^2], \quad (\text{C6})$$

we may worry that this procedure will fail to give back the original CFT, due to relevant terms generated by integrating out  $B$  and then  $A$ . For Eq. C4, where the order of limits is the opposite, it is even less clear that we will obtain  $S^2 = -1$ . In general this is a nontrivial question about the structure of the RG flows.

If the gauged  $\text{CFT}_1$  does indeed flow to a nontrivial  $\text{CFT}_2$ , then characteristic lengthscale for this crossover is  $l_e \sim 1/e^2$ . Gauging  $\text{CFT}_2$  then introduces a new lengthscale  $l_{e'} \sim 1/e'^2$ , and the regime of interest is  $l_{e'} \gg l_e$ . In order for  $S^2 = -1$  to hold, the ultimate flow on scales  $\gg l_{e'}$  must be to a copy of the  $\text{CFT}_1$  fixed point.

If the gauged  $\text{CFT}_1$  instead flows to a trivial theory, representing for example a massive or symmetry-broken fixed point, then it is hard to see how  $S^2 = -1$  can ever be satisfied. Here we are assuming that  $\text{CFT}_1$  is nontrivial; it is certainly possible to have an example where  $\text{CFT}_1$  and  $\text{CFT}_2$  are both trivial, and  $S^2 = -1$ .<sup>56</sup>

This discussion is intended to provide intuition for dangers which may arise in the formal use of the  $S$  operation. It is of course conceivable that in practise they do not arise. We should also emphasise that the limiting procedure discussed above is not the only way to interpret the formal definition of  $Z_S$ : it is possible that when the above limits fail to give a nontrivial CFT, the  $S$  operation can be rescued by an alternative implementation of the definition.

<sup>56</sup> An instructive example is to take  $\text{CFT}_1$  to be a trivial theory with partition function  $Z_{\text{CFT}_1} = 1$ . In a condensed matter context this is the fixed point theory for a bosonic Mott insulator. If we now couple the bosons to a dynamical gauge field to obtain  $\tilde{Z}_S$  we see that the new theory is in a phase where the new global  $U(1)$  symmetry associated with conservation of  $dA$  is spontaneously broken (a superfluid). If we use the formal definition in Eq. C1 we get  $Z_S[B] = \delta(B)$  as expected for a superfluid. In a formal sense this partition function is conformally invariant — we may think of it as the fixed point description of a superfluid phase obtained by taking the phase stiffness of the Goldstone mode to  $\infty$ . Of course  $Z_S$  is physically lacking as it misses the Goldstone physics.

If we ignore all the caveats and *assume* (as is normally done in the literature) that there is a well-defined  $SL(2, Z)$  action on 2 + 1D CFTs then we can make some powerful statements. First it tells us that there is a CFT which looks like easy plane  $NCCP^1$ , defined by the partition function

$$\int \mathcal{D}b (Z_{\text{WF}}[b])^2 e^{i \int \frac{1}{2\pi} b dB} \quad (\text{C7})$$

where  $Z_{\text{WF}}[b]$  is the partition function for the Wilson Fisher fixed point of a single complex boson, with background gauge field  $b$ . A priori we do not know whether the standard easy plane  $NCCP^1$  action (defined with, e.g., an additional Maxwell term) flows to this CFT.

Second, if we assume that the formal  $S$  operation gives a well defined action on CFTs, then it is natural to expect that basic boson-fermion duality, relating a single massless Dirac fermion to a Wilson-Fisher boson coupled to a  $U(1)_1$  gauge field, can be taken as an exact statement about path integrals:

$$Z_D[A] = \int \mathcal{D}b Z_{\text{WF}}[b] e^{i \int d^3x \frac{1}{4\pi} b db + \frac{1}{2\pi} b dA} \quad (\text{C8})$$

where  $Z_D$  is the partition function of a free massless Dirac fermion. We write this as (recall that  $T$  shifts the level of the Chern Simons term)

$$D = ST[\text{WF}]. \quad (\text{C9})$$

The other boson-fermion duality then is

$$D = T^{-1}S^{-1}T^{-1}[\text{WF}]. \quad (\text{C10})$$

Multiplying the partition functions on both sides, shifting an  $AdA/4\pi$  to the left, and finally making  $A$  dynamical we get the duality of  $\text{QED}_3$  to the easy plane CFT defined in (C7).

The fermion side is manifestly  $[\text{SU}(2) \times U(1)]/Z_2$  invariant. Further it is easy to see that it is exactly self-dual and the dual side has the other  $[\text{SU}(2) \times U(1)]/Z_2$  as a manifest symmetry. Altogether this implies the  $O(4)$  symmetry.

Within the framework of the present assumptions, these are all exact statements, regardless of the relevance or irrelevance of operators that break  $O(4)$  to  $\text{SU}(2) \times U(1)$  or  $U(1) \times U(1)$ . As mentioned above, it is possible that the fixed point we are describing is highly fine-tuned. But if we now make the natural *further* assumption that the weak coupling limits of all these gauge theories flow to the IR CFTs defined formally but exactly by the path integrals above, and that there is no fine tuning hidden in this flow, we indeed conclude that the various symmetry-allowed perturbations are irrelevant. However we emphasize again that this view on the dualities and emergent symmetries

is predicated on the reliability of the formal  $SL(2, Z)$  action on 2 + 1D CFTs which as far as we are aware still remains conjectural.

### Appendix D: Some useful mathematical concepts

It is convenient to consider a “triangulation” of the space-time manifold  $M$  (we will be mostly interested in manifolds of dimension  $D = 4$ ) : we represent points in space-time by a discrete lattice where each elementary unit is a  $D$ -simplex. Pick a local ordering of the vertices of the lattice. A  $k$ -cochain lives on  $k$ -simplices, i.e. it is a function that depends on  $(k + 1)$  vertices and takes values in some abelian group  $G$ . We will only need to consider the cases  $Z$ ,  $Z_2$  and  $Z_4$ . The corresponding cochain is then said to be an element of  $C^k(M, G)$ . For instance a 2-cochain in  $C^2(M, Z_2)$  is a function  $a_{ijk} = 0, 1$  while for a 2-cochain in  $C^2(M, Z)$ , the function  $a_{ijk} \in Z$ . Here  $(ijk)$  are the vertices of a triangular plaquette of the simplex.

We can define a discrete derivative (known as a “coboundary”) operation  $d$  that maps  $k$ -cochains to  $(k + 1)$ -cochains:

$$(da)_{(i_0, i_1, i_2, \dots, i_{k+1})} = \sum_{p=0}^{k+1} (-1)^p a_{i_0, i_1, \dots, \widehat{i_p}, \dots, i_{k+1}} \quad (D1)$$

where the variable  $\widehat{i_p}$  is omitted. It is understood that the addition on the right side is performed in  $G$  (e.g. mod 2 addition for  $G = Z_2$ ). It is readily checked that  $d^2a = 0$ . The set of all  $k$ -cochains  $a$  that satisfy  $da = 0$  form a group under addition known as the cocycle group  $Z^k(M, G)$ . The set of all  $a \in C^k(M, G)$  that may be written  $a = db$  for some  $b \in C^{(k-1)}(M, G)$  form a different group known as the coboundary group  $B^k(M, G)$ . Clearly  $B^k(M, G) \subset Z^k(M, G)$ . The cohomology group  $H^k(M, G) = \frac{Z^k(M, G)}{B^k(M, G)}$ .

For two 2-cochains  $a \in C^k(M, G)$  and  $b \in C^l(M, G)$ , we define the cup product

$$(a \cup b)_{i_0, \dots, i_{k+l}} = a_{i_0 i_1 \dots i_k} b_{i_k i_{k+1} \dots i_{k+l}} \quad (D2)$$

where  $i_0, \dots, i_{k+l}$  are assumed to be ordered. The cup product satisfies

$$d(a \cup b) = da \cup b + (-1)^k a \cup db. \quad (D3)$$

Clearly if  $da = 0 = db$  then  $d(a \cup b) = 0$ . Thus the cup product defines a product of cohomology classes. The cup product is a generalization of the familiar wedge product of differential forms.

The Poyntryagin square of  $w \in H^2(M, Z_2)$  plays an important role in our discussion (please see Ref. [109] for more discussion and references). We now specialize to 4-manifolds  $M = Y_4$ . It is

easiest to define if  $w$  can be lifted to an element  $\widehat{w} \in H^2(Y_4, Z)$ , i.e.  $w = \widehat{w} \pmod{2}$  and  $d\widehat{w} = 0$ . In this case,  $\mathcal{P}(w) = \widehat{w} \cup \widehat{w} \pmod{4}$ . If  $w$  does not admit a lift to an integral cohomology class then  $\mathcal{P}$  is still a  $\pmod{4}$  quantity. It is defined to be

$$\mathcal{P}(w) = w \cup w + w \cup_1 dw \pmod{4} \quad (\text{D4})$$

The new product  $\cup_1$  is defined (for a 2-cochain  $a$  and a 3-cochain  $b$ ) as

$$(a \cup_1 b)_{01234} = a_{034}b_{0123} + a_{014}b_{1234} \quad (\text{D5})$$

It is readily seen that  $\mathcal{P}(w)$  transforms by a co-boundary under  $w \rightarrow w + 2n$ ,  $w \rightarrow w + dm$ , so it is well-defined on  $H^2(Y_4, Z_2)$ . Note that as  $w \in H^2(Y_4, Z_2)$ ,  $dw = 0 \pmod{2}$ . Thus we have  $\mathcal{P}(w) = w \cup w \pmod{2}$ . It can be shown that

$$\mathcal{P}(w + w') = \mathcal{P}(w) + \mathcal{P}(w') + 2w \cup w' \pmod{4} \quad (\text{D6})$$

We will use this repeatedly.

### Appendix E: Topological superconductors and the APS theorem

In this appendix we review the field-theoretic description of topological superconductors in  $2+1$  and  $3+1$ D. We follow Ref. [54] here and adapt it to the  $\text{SO}(n)$  symmetric systems of interest in this paper.

Let's begin with  $2+1$ D. A  $p_x + ip_y$  superconductor can be represented by a massive 2-component Majorana fermion  $\chi$ :

$$L = \bar{\chi}(D + m)\chi \quad (\text{E1})$$

with  $\bar{\chi} = \chi^T C^*$ ,  $D = \gamma^\mu(\partial_\mu + i\omega_\mu)$ , and  $C = \sigma^y$  - the charge-conjugation matrix.  $m > 0$  corresponds to the trivial superconductor and  $m < 0$  to the  $p_x + ip_y$  superconductor. The point  $m = 0$  corresponds to the transition between these two phases. Now, the formal partition function of (E1) on a closed manifold  $M$  is

$$Z(m) = \text{Pf}(C^\dagger(D + m)) = \pm \det(D + m)^{1/2} = \pm \prod_{\langle \lambda \rangle} (-i\lambda + m)^{N(\lambda)/2} \quad (\text{E2})$$

where the product is over eigenvalues  $\lambda$  of the Dirac operator  $iD$  (without repetitions) and  $N(\lambda)$  is the multiplicity of the eigenvalue. Since  $[CK, iD] = 0$  and  $(CK)^2 = -1$ , all eigenvalues of  $iD$  are

doubly degenerate. The above expression clearly requires regularization. We note that the partition function of the trivial superconductor at long wavelength (or equivalently in the  $m \rightarrow \infty$  limit) is expected to be analytic in the curvature of the manifold and to have no topological terms, so it can be effectively set to 1. It is then convenient to normalize other partitions functions by it. This can be understood as the physical justification of Pauli-Villars regularization. Then,

$$Z(m)_{PV,+} = \lim_{M \rightarrow \infty} \frac{Z_m}{Z_M} = \prod_{\langle \lambda \rangle} \frac{(-i\lambda + m)^{N(\lambda)/2}}{(-i\lambda + |M|)^{N(\lambda)/2}} \quad (\text{E3})$$

Note that there is no sign ambiguity in Eq. (E3). Indeed, we can reach any value of  $m$  starting with the trivial insulator at  $m = \infty$ . The requirement that the partition function during this process be analytic in  $m$  removes the sign ambiguity. Now, when  $m = 0$ , we can write the partition function (E3) as  $Z = |Z|e^{i\varphi}$ , where the phase

$$\varphi = -\frac{1}{2} \sum_{\langle \lambda \neq 0 \rangle} N(\lambda) \text{sgn}(\lambda) \tan^{-1} \frac{|M|}{|\lambda|} \rightarrow -\frac{\pi}{4} \sum_{\lambda \neq 0} \text{sgn}(\lambda) \quad (\text{E4})$$

The sum in the last term is over all eigenvalues of  $iD$  (repeated eigenvalues included) and we've taken the  $M \rightarrow \infty$  limit naively. While the resulting final sum is formal, it can be equivalently regulated with the  $\zeta$  function method, giving

$$Z(m=0)_{PV,+} = |Z(m=0)| \exp(-\pi i \eta(iD)/4) \quad (\text{E5})$$

with  $\eta$  defined via Eq. (116).<sup>57</sup> Deep in the  $p_x + ip_y$  phase, we may set  $m = -|M| \rightarrow -\infty$  in Eq. (E3) and obtain a pure phase,

$$Z_{p_x+ip_y} = \exp(-\pi i \eta(iD)/2) \quad (\text{E6})$$

The APS theorem (120) allows us to rewrite the partition function of a  $p_x + ip_y$  superconductor, Eq. (E6), as

$$Z_{p_x+ip_y} = \exp(-iCS_g[Y_4]) \quad (\text{E7})$$

where, as explained in the section VII, the gravitational Chern-Simons term  $CS_g$  (119) is defined via a continuation of  $M$  to an auxiliary four-manifold  $Y_4$ . The APS theorem guarantees that the result is independent of the continuation. The gravitational Chern-Simons term encodes precisely

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<sup>57</sup> Note that  $\eta$  in (116) includes a contribution  $N_0$  from the zero modes  $\lambda = 0$ . If there are any zero modes then the absolute value of the partition function,  $|Z(m=0)|$ , vanishes and the phase is irrelevant.

the thermal-Hall response of a  $p_x + ip_y$  superconductor:  $\frac{\kappa_{xy}}{T} = \frac{1}{2}$ . Note that our Majorana fermions  $\chi$  require a spin structure, and the continuation of  $M$  to  $Y_4$  must preserve this spin structure. Thus,  $CS_g[Y_4]$  is secretly spin-structure dependent (as, less surprisingly, is the  $\eta$  invariant).

We can easily generalize the above discussion to  $n$  identical Majorana fermions. Now we may couple the system to an  $SO(n)$  gauge field  $A$ , so that the Dirac operator reads  $D_A = \gamma^\mu(\partial_\mu + i\omega_\mu - iA_\mu)$ .<sup>58</sup> The partition function for  $m \rightarrow -\infty$  (i.e. deep in the topological phase) again is (E6) with  $\eta(iD)$  now referring to the full Dirac operator  $D_A$ . This can be rewritten using the APS theorem (120) as,

$$Z_{(p_x+ip_y)^n} = \exp\left(-i(CS_{SO(n)}[A, Y_4] + nCS_g[Y_4])\right) \quad (\text{E8})$$

with the  $SO(n)$  Chern-Simons term, given by Eq. (119), again defined via the continuation to  $Y_4$ . Thus,  $n$ -copies of a  $p_x + ip_y$  superconductor have  $SO(n)$  response characterized by a Chern-Simons term at level 1. For  $n = 2$  this simply corresponds to  $\sigma_{xy} = 1$ . There is also the expected thermal Hall response  $\frac{\kappa_{xy}}{T} = n \times \frac{1}{2}$  encoded in the gravitational Chern-Simons term. As emphasized in section VII, for even  $n$  the Majorana fermions really see an  $(SO(n) \times Spin(3)_{TM})/Z_2$  bundle, i.e. we don't need to separately specify the spin structure, but only the combination of  $SO(n)$  and  $Spin(3)_{TM}$  transition functions. Likewise, on  $Y_4$  we again need to continue just the  $(SO(n) \times Spin(3)_{TM})/Z_2$  bundle. Finally, the partition function for  $m = 0$  (the transition point between a trivial phase and  $n$  copies of a  $p_x + ip_y$  superconductor) is again given by Eq. (E5).

Now, let us proceed to 3 + 1D. A topological superconductor in class DIII can be represented by a massive (four-component) Majorana fermion. For generality, we work from the start with  $n$  identical copies of a topological superconductor and couple the system to an  $SO(n)$  gauge field  $A$ , so the continuum bulk action is:

$$L = \bar{\chi}(D_A + m)\chi \quad (\text{E9})$$

where again  $D_A = \gamma^\mu(\partial_\mu + i\omega - iA_\mu)$ ,  $\bar{\chi} = \chi^T C^*$  and  $C$  is the charge-conjugation matrix. The phase with  $m > 0$  may be taken (by convention) to represent the trivial superconductor and  $m < 0$  - the topological superconductor. As before, we may set the partition function of the trivial phase,  $m \rightarrow \infty$ , to 1, so the bulk partition function of the topological phase,  $m \rightarrow -\infty$ , on a closed four-manifold  $X_4$  is:

$$Z_{TSc} = \prod_{\langle \lambda \rangle} \frac{(-i\lambda - |M|)^{N(\lambda)/2}}{(-i\lambda + |M|)^{N(\lambda)/2}} \quad (\text{E10})$$

<sup>58</sup>  $iA_\mu$  is real, so crucially,  $[iD_A, CK] = 0$  still holds.

with  $\lambda$  - eigenvalues of  $iD_A$ . Crucially, again  $[CK, iD_A] = 0$  and  $(CK)^2 = -1$ , so all eigenvalues are doubly degenerate. What is different compared to the previously discussed  $p_x + ip_y$  case, is that  $\{\gamma^5, iD_A\} = 0$ , so all non-zero eigenvalues of  $iD_A$  come in pairs  $\pm\lambda$ ; therefore, their contribution to the partition function cancels and

$$Z_{TSc} = (-1)^{N_0/2} \quad (\text{E11})$$

where  $N_0$  is the number of zero modes of  $iD_A$ . The zero modes can be chosen to be simultaneous eigenstates of  $\gamma^5$ . Suppose there are  $N_{\pm}$  eigenstates with  $\gamma^5 = \pm 1$ . We note that  $N_+$  and  $N_-$  are separately even as  $[CK, \gamma^5] = 0$ . Therefore, we may rewrite  $Z_{TSc} = (-1)^{(N_+ - N_-)/2}$ . The difference  $N_+ - N_- = 2\mathcal{J}$  is known as the index of  $iD_A$  (we include a prefactor of 2 to emphasize that in the present situation it is even), and we may write:

$$Z_{TSc} = (-1)^{\mathcal{J}} \quad (\text{E12})$$

We see that the partition function is real, as it should be for a time-reversal invariant system on an orientable manifold. The Atiyah-Singer theorem[77, 78], tells us that

$$2\mathcal{J} = N_+ - N_- = \frac{1}{\pi}(CS_{SO(n)}[A, X_4] + nCS_g[X_4]) = p_1[A, X_4] - \frac{n\sigma[X_4]}{8} \quad (\text{E13})$$

with the Pontryagin number  $p_1$  and signature  $\sigma$  given by Eqs. (124), (125), so that we may re-write

$$Z_{TSc} = \exp \left[ \frac{i}{2}(CS_{SO(n)}[A, X_4] + nCS_g[X_4]) \right] \quad (\text{E14})$$

When  $n = 1$  (or more generally for odd  $n$ ), we must pick a spin-structure for our fermions  $\chi$  (in particular,  $X_4$  must admit a spin structure); we then learn from (E13) that on a spin manifold  $\sigma$  is a multiple of 16. Furthermore, if we fix a spin-structure,  $A$  is a true  $SO(n)$  gauge field (with transition functions satisfying the cocycle condition), from which we learn that  $p_1[A]$  on a spin manifold is even. Now, for even  $n$ , we don't require  $X_4$  to admit a spin structure: the fermions see transition functions in  $(SO(n) \times Spin(4)_{TX_4})/Z_2$  group, so while  $p_1$  and  $\frac{n\sigma}{8}$  themselves need not be even (in fact,  $\sigma$  is an integer, so  $\frac{n\sigma}{8}$  is generally a fraction), the combination  $p_1 - \frac{n\sigma}{8}$  is an even integer.

Finally, let us discuss the case when the topological superconductor lives on a space  $X_4$  with a boundary  $M$ . We know that the boundary supports  $n$  gapless Majorana cones. The bulk+boundary partition function now is[54]:

$$S_{TSc}^{\text{bulk+bound}} = \int_M [\bar{\chi} D_A \chi]_{PV,+} - \frac{i}{2}(CS_{SO(n)}[A, X_4] + nCS_g[X_4]) \quad (\text{E15})$$

The Majorana action (E15) depends only on the boundary data; on the other hand, the second term in Eq. (E15) depends on the bulk, and in-fact, reduces to our previous expression (E14) for a closed manifold. While each term in Eq. (E15) is separately well defined, the time-reversal symmetry of (E15) is not obvious. However, using (E5) and the APS theorem (120), we obtain

$$Z_{TSc}^{\text{bulk+bound}} = |Z^{\text{bound}}(m=0)|(-1)^{\mathcal{J}[A,X_4]} \quad (\text{E16})$$

where  $|Z^{\text{bound}}(m=0)|$  is the absolute value of boundary Majorana fermion partition function, and  $2\mathcal{J}$  is the index of the bulk Dirac operator  $iD_A$  with APS boundary conditions. The time-reversal symmetry is now manifest.

To obtain further physical intuition for the action (E15), we may break time-reversal symmetry on the surface with a mass term  $m\bar{\chi}\chi$ . Integrating the Majorana fermions out, by our preceding discussion we then obtain at long wave-length

$$S = \mp \frac{i\text{sgn}(m)}{2} (CS_{SO(n)}[A, X_4] + nCS_g[X_4]) \quad (\text{E17})$$

i.e. the surface has  $SO(n)$  Chern-Simons response at level  $1/2$  and thermal-Hall response with  $\kappa_{xy}/T = n \times \frac{1}{4}$ . This is precisely what we expect for the  $T$ -broken surface state of  $n$ -copies of a topological superconductor.

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